Quantitative Economics 5 (2014), 705–749 1759-7331/20140705

Altruistically motivated transfers under uncertainty

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How do families behave dynamically? We provide a framework for studying economic problems in which family behavior is essential. Our key innovation is the inclusion of imperfectly altruistic agents in an otherwise standard consumption–savings problem with exogenous income risk. This gives rise to altruistic transfers and strategic behavior in the consumption–savings decision. We study the Markov-perfect equilibrium that arises from the limit of equilibria in a sequence of finite games. The equilibrium's transfer patterns are empirically plausible. Furthermore, agents overconsume relative to the social optimum. In contrast to two-period models, *both* the richer and the poorer players overconsume long before transfers actually occur. The poorer agent also faces incentives to engage in excessive risk-taking because losses from a gamble are absorbed by both while gains are enjoyed alone.

KEYWORDS. Altruism, inter vivos transfers, consumption–savings decision, differential games.

JEL CLASSIFICATION. C73, D1, D64, E21.

1. INTRODUCTION

How do agents within a family behave dynamically? How do they make consumption and savings decisions when they know that they can count on transfers from other family members or that they will provide transfers to others themselves? How do expectations about these transfers affect risk-taking behavior? Answering these questions is clearly important per se. Furthermore, for policy analysis, the response of families has to be taken into account in many areas.

This paper studies how agents within a family interact dynamically. We aim to provide a step toward the objective of developing a theory that can be used in quantitative

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For useful discussions and comments, we thank Luisa Fuster, Nezih Guner, Jonathan Heathcote, Greg Kaplan, Dirk Krueger, Per Krusell, Victor Ríos-Rull, Tom Sargent, Kjetil Storesletten, Gianluca Violante, Ludo Visschers, several anonymous referees, and seminar and conference participants at the EEA-ESEM Conference 2011, INSEAD, McGill University, the Minnesota Federal Reserve Bank, Queen Mary, the REDg 2011 Workshop, and University of Southampton. Matthias Kredler acknowledges research funding from the Spanish Ministerio de Ciencia e Innovación, reference SEJ2007-62908, and from the Spanish Ministerio de Economía y Competitividad, reference ECO2012-34581.

DOI: 10.3982/QE353

macroeconomics. Our framework stays as close as possible to the Bewley model, which is the baseline framework used in the macroeconomic literature to model heterogeneous agents and the implications of uninsured idiosyncratic risk. The main difference is that we study families consisting of two members who are imperfectly altruistic toward each other, commitment being absent. This setting gives rise to altruistic transfers and strategic behavior in the consumption–savings decision. When altruism is perfect or absent, our model collapses to a standard Bewley economy.

We formulate the dynamic game in a continuous-time setting and characterize the Markov-perfect equilibrium that arises from the limit of equilibria in a sequence of finite games (an equilibrium selection criterion previously used by Klein, Krusell, and Ríos-Rull (2008), among others). Continuous time enormously simplifies the characterization of equilibria by making consumption and savings decisions independent of the contemporaneous choices of the other player. In our continuous-time game, agents' best response functions over an infinitesimal amount of time are constant. Nonetheless, the strategy of the other player affects savings and transfers since it affects the continuation value of the game and, hence, the current value of savings.

Our analysis sheds light on strategic interactions between altruistic agents. There is a crucial difference between strategic transfers in our no-commitment setting and transfers in a commitment setting, as previously studied by Altig and Davis (1988, 1992, 1993), for example. Under commitment, the timing of transfers is indeterminate, while our setting delivers clear predictions in this respect. We find that strategic transfers only flow when one of the agents is facing a binding borrowing constraint. This strategic delay of transfers is explained by the fact that the equilibrium features overconsumption relative to the social optimum.[1] The poor agent behaves recklessly by overconsuming, counting on the benevolence of the richer agent. To minimize the incentives for overconsumption, the donor only makes transfers when the receiving agent is constrained. Anticipating the reckless behavior of the poor agent, the rich agent also has incentives to overconsume relative to the first-best. The equilibrium thus features overconsumption by both agents who engage in a race to the bottom. This inefficiency resembles the tragedy of the commons: the assets of the rich agents are a common resource that is being depleted by both agents. We call this result the *dynamic Samaritan's dilemma*. This extends what Lindbeck and Weibull (1988), Bernheim and Stark (1988), and Bruce and Waldman (1990) refer to as the *Samaritan's dilemma*: in a two-period model, the donor delays transfers to the second period and the recipient's first-period savings are inefficiently low. Tracing the effects further back in time, we show that *both* agents' savings decisions are distorted (not only those of the recipient) and that these distortions occur long before transfers actually flow.

Furthermore, when introducing a portfolio choice into our setting, we find that the poorer agent may engage in excessive risk-taking. This occurs because some of the downside risk is borne by the future donor while upside risk is enjoyed alone. The excessive risk-taking mirrors a result by Laitner (1988), who finds that risk-averse imperfectly

[1]The result that transfers only flow to constrained recipients confirms a conjecture previously made by papers such as Fuster, Imrohoroglu, and Imrohoroglu (2007), Laitner (2001), McGarry (1999), and Nishiyama (2002). In our paper, this is an endogenous outcome.

altruistic family members effectively become risk-loving and make use of fair lotteries when given this option. While in his framework, generations overlap for only one period, our generations overlap for many periods. Our framework is thus better suited to study risk-taking behavior quantitatively; examples for such behavior are risky career choices, purchasing insufficient insurance, or risky portfolio choices.

The model's predictions on transfers are in line with empirical findings. In the model, as in the data, transfers flow from wealth- and income-rich agents to poor recipients. The size of transfers is increasing in the donor's wealth and labor income, and decreasing in the recipient's labor income (see McGarry and Schoeni (1995, 1997) and Berry (2008)). Donors in the model only give transfers to borrowing-constrained recipients so as to keep recipients' consumption spending in check. Indeed, Cox (1990) and Cox and Jappelli (1990) find that transfers flow primarily to liquidity-constrained individuals. Finally, agents in our model have a strong strategic incentive to delay transfers so as to maintain control over the allocation. This is consistent with the fact that a large share of intergenerational transfers flows in the form of bequests, although this is discouraged by estate taxes in most countries.

Finally, to facilitate applying our model in a broader context, we show in the Computational Appendix—available in a supplementary file on the journal website, http://qeconomics.org/supp/353/supplement.pdf—how our solution algorithm can be extended to more complex environments, such as overlapping-generations settings, finite-horizon settings, and settings with time- or age-dependent value functions. We also provide an introduction to the Markov-chain approximation method for continuous-time problems, which is not used much in the economics literature. A code toolbox that can be used to replicate all the results presented in the paper and for use in future research is available in a supplementary file on the journal website, http://qeconomics.org/supp/353/code_and_data.zip.

Our model is applicable to all areas in which family transfers are an important source of insurance. Families may provide monetary transfers to a member in hardship (e.g., to an unemployed member, to relieve borrowing constraints, remittances, and support to an elderly), to help a child with financing educational expenses or the purchase of a home, and in the form of leaving a bequest. Families may also provide nonmonetary transfers such as in-kind transfers (providing a good instead of cash), time transfers (e.g., child care or informal care to the frail elderly), or spatial transfers (e.g., when a young adult moves back home in times of high youth unemployment or when a frail elder moves in with her child to avoid institutionalization in a nursing home). If a government becomes active in such areas (through unemployment insurance, pension and welfare payments, deficit-financed tax cuts, student loans, home-buyer credit programs, estate tax, long-term-care policies such as subsidizing informal caregivers or nursing homes, child-care subsidies, or subsidies for housing), families' behavioral response has to be taken into account so as to predict the policy's outcomes and welfare implications.

The standard workhorse models in macroeconomics are often ill-suited to address the interplay between family behavior and policy. These models make one of two extreme assumptions about household linkages within the extended family (or dynasty).

Infinitely-lived-agent models implicitly assume that altruism within the dynasty is perfect. Overlapping-generations (OLG) models, in contrast, assume that agents' altruism toward other generations in the extended family is absent.

Modeling a family as an infinitely lived agent has the problem that there is no notion of wealth positions of the different generations. Consequently, there are no predictions on the timing of transfers. Closely related to the infinitely-lived-agent framework are models in which imperfectly altruistic family members can commit to allocations.[2] Again, these models have no predictions on the distribution of wealth and the timing of transfers. This is because commitment implies that it does not matter which family member carries along assets. For example, a rich parent in such models is typically indifferent between signing over her entire wealth to her children before dying and the alternative of leaving it as a bequest. In reality, however, the parent may have strong incentives to hold on to her assets so as to stay in control of her situation.

In OLG models, the problem is reversed. These have clear predictions on the wealth positions of subsequent generations, but no transfers between generations occur (maybe barring accidental bequests).[3]

Barczyk (2012) provides an illustration of where our setting stands compared to the two benchmark models. He considers the transition dynamics of aggregate consumption in response to a deficit-financed tax cut using our framework and compares it to an OLG and a dynastic economy. Interestingly, the transition dynamics of aggregate consumption with imperfect altruism is not simply a convex combination of the OLG economy and the dynastic economy. Instead, the response in aggregate consumption often actually *exceeds* the one in the OLG economy. Welfare implications are, however, closer to those from a dynastic economy.

In our model, a family/dynasty consists of two decision makers. These agents can represent an old household and a young household within a family, but, depending on the context, we could also think of them as other entities, for example, spouses with separate bank accounts or countries that give development aid to each other. When we formally introduce the model in Section 2, we will refer to the two decision makers simply as "he" and "she," and leave the interpretation to the reader. Our setting nests both the infinitely lived agent and the OLG model as special cases.

While we are obviously not the first to study a model with altruism, our major innovation is that we allow for a flexible degree of altruism in a setting where agents overlap for many periods. Agents cannot commit to future actions, which makes the model

[2]In Altig and Davis (1988, 1992, 1993), for example, generations can commit to future transfer payments. In the so-called *collective* model, family members can commit to a fully contingent plan for all choice variables (see Chiappori (1988)).

[3]Adding a warm-glow motive for giving bequests to subsequent generations does not change the fundamental properties of the OLG model when it comes to intergenerational redistribution. With warm glow, parents care about leaving resources to their children. But they regard bequests as a good, that is, they derive utility from the size of the bequest regardless of how much their children need this bequest. This implies that parents' bequests do not react to policies that make their children worse off, which is incompatible with altruistic preferences à la Becker.

suited for studying environments where it is reasonable to believe that contracts are too costly or impossible to enforce.[4]

Our innovation is important and challenging. Laitner (1988) and Fuster, Imrohoroglu, and Imrohoroglu (2007), for example, explicitly call for an exploration of a multiperiod model with imperfectly altruistic agents and no-commitment so as to study theoretically and quantitatively more realistic environments. Some recent applied papers that are concerned with family transfers also have made recommendations in this direction. Mazzocco (2007) studies a version of the collective model in which the commitment assumption is relaxed and finds support for this model in the data. Our model can be seen as a different, noncooperative approach to relaxing the commitment assumption. From an empirical perspective, McGarry (2006) argues that "[...] evidence suggests that dynamic models can provide insights into transfer behavior that are impossible to obtain in a static context." Finally, Kaplan (2012) studies a model of young workers who have the option to co-reside with their parents and finds that a noncooperative approach without commitment is a preferable modeling choice over a setting with commitment in view of the data. In his setting, however, parents cannot save. The author acknowledges that this assumption is not innocuous, but remarks on the technical and computational difficulty that a fully strategic dynamic setting would bring with it.

Laitner (1988) analyzes an OLG model with imperfect altruism and no-commitment in which generations overlap for one period. This provides substantial tractability, but limits the scope of transfer behavior. The drawback is the implicit assumption that agents can commit to a lump-sum transfer at the beginning of a life-cycle stage. Since one stage represents roughly 25 years, this is not innocuous. Consequently, the model also lacks precise predictions on the timing of transfers. Our innovations allow for a finer time resolution and thus more realistic calibrations, especially when it comes to studying short-term effects of policies and transitional dynamics (see Barczyk (2012), for an example). Related to this, our model has a richer scope of transfer behavior. Transfers can flow temporarily, never, or always.

A framework with altruism in which households overlap for many periods can also be made tractable by assuming perfect two-sided altruism (the dynastic model) or imperfect altruism with commitment. Examples of the former approach include Laitner (1992, 1993), Fuster (1999), Fuster, Imrohoroglu, and Imrohoroglu (2003, 2007), and Heathcote (2005); examples of the latter approach are Altig and Davis (1988, 1992, 1993). These assumptions do provide substantial tractability, but come at a cost: the models generate too many inter vivos transfers and a relatively low incidence of binding liquidity constraints (see Fuster, Imrohoroglu, and Imrohoroglu (2007) and Laitner (1993)). Furthermore, as mentioned above, these models have no predictions on the distribution of wealth within the family and the timing of transfers.

[4]There are some two-period models in the literature in which strategic interactions occur (i.e., there is no commitment) (e.g., Lindbeck and Weibull (1988)). However, these models make simplifying assumptions in that they restrict transfers to flow in certain situations and focus on specific timing protocols. This limits their usefulness for extension.

Finally, Barczyk and Kredler (2014) study a cake-eating problem with imperfect altruism in a deterministic setting without commitment. While their results guide our analysis in important ways, there are crucial differences in results. They find a continuum of equilibria in which imperfectly altruistic agents act as if they were a perfectly altruistic dynasty in the long run. However, there are multiple indeterminacies in this class of equilibria that makes their framework hard to apply in practice. The current paper delivers a unique equilibrium (under certain restrictions) that can be readily computed in a large range of variations of the basic setting.

The remainder of the paper is organized as follows. Section 2 provides the setting and the equilibrium definition. This is followed in Section 3 by an analysis of the incentives the players face in the cases when they are unconstrained and constrained. Section 4 presents our main results. Testable implications of the model are derived in Section 5. Section 6 concludes.

2. Setting

2.1 *Physical environment*

Time t is continuous. There are two infinitely lived players referred to as "she" and "he." We will denote variables pertaining to her as plain lowercase letters (e.g., c) and variables pertaining to him with primed letters (e.g., c').

Agents obtain an exogenous endowment stream $\{y, y'\}$. The endowment streams follow independent Poisson processes with common support $y, y' \in \{y_1, y_2\}$, where $y_1 < y_2$. The Poisson rates of transitioning from high to low and low to high income are ξ for both players.[5] Agents can save in an asset with return r and are subject to a no-borrowing constraint. Agents' wealth position is subject to independent shocks with standard deviation σ.[6]

At each point in time, agents choose a consumption rate, $c_t \geq 0$, and a nonnegative transfer rate, $g_t \geq 0$, to the other agent (g stands for "gift"). These choices imply the laws of motions for wealth,

$$\mathrm{d}w_t = \underbrace{\left(rw_t + y_t - c_t - g_t + g'_t\right)}_{\equiv \dot{w}_t\text{: her savings rate}} \mathrm{d}t + w_t \sigma \, \mathrm{d}B_t, \tag{1}$$

$$\mathrm{d}w'_t = \underbrace{\left(rw'_t + y'_t - c'_t - g'_t + g_t\right)}_{\equiv \dot{w}'_t\text{: his savings rate}} \mathrm{d}t + w'_t \sigma \, \mathrm{d}B'_t, \tag{2}$$

where w stands for wealth, and B_t and B'_t are uncorrelated standard Brownian motions. When $w = 0$, we require $c + g \leq y$ (and equivalently for him). Also, note that when $w = 0$, the Brownian motion does not enter the law of motion.

[5]A Poisson process is the analog to a Markov process in discrete time. It is straightforward to generalize our results to general Poisson processes. We focus on this simple case for expositional ease only.

[6]We add Brownian motion to the law of motion as a randomization device to ensure existence and uniqueness of an equilibrium in pure strategies in the limit of finite games. We will discuss this assumption and its implications in detail in Section 4.3.

Her preferences are represented by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t}\left[u(c_t) + \alpha u(c_t')\right] \mathrm{d}t.$$

The discount rate is $\rho > 0$ and $\alpha \in [0, 1]$ is the parameter that measures the intensity of altruism. He is a mirror-symmetric copy of her, but may have a different altruism parameter $\alpha' \in [0, 1]$ from hers. His preferences are represented by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t}\left[u(c_t') + \alpha' u(c_t)\right] \mathrm{d}t.$$

We assume that agents do not differ in their discount rates and in the form of the felicity function $u(\cdot)$. This formulation encompasses the case of selfishness ($\alpha = \alpha' = 0$), perfect altruism ($\alpha = \alpha' = 1$), and one-sided altruism ($\alpha > 0$, $\alpha' = 0$). For perfect altruism, the players' objectives are perfectly aligned and the model collapses to a single-agent model. For selfishness, the model becomes a standard consumption–savings model with two agents (or an OLG model once a demographic structure is added).

2.2 *Equilibrium definition*

We focus on Markov-perfect equilibria and study equilibria that are obtained as the limit of a sequence of finite games (see Klein, Krusell, and Ríos-Rull (2008), for a discussion of this equilibrium-selection criterion). The payoff-relevant state is given by $x \equiv (w, w', y, y')$. A Markovian strategy is a pair of nonnegative functions $\{c(x), g(x)\}$ for her and a pair $\{c'(x), g'(x)\}$ for him.[7]

Given the continuous-time setting, agents can choose arbitrarily high c and g whenever they have positive wealth. After all, there always exists a time horizon Δt that is short enough such that any policy can be maintained over Δt. This is not true any more, however, once wealth is zero and the borrowing limit is reached. Feasibility of a plan then depends on the transfers the other player is giving. Since strategies can only depend on the state x but not on the other player's action, we draw a distinction between an agent's strategy c and "realized consumption" c^*, which we define as an outcome of the game. Agents' preferences are defined over the outcome c^*. We define $c^* = c$ whenever $w > 0$, but specify realized consumption when she has run out of assets as

$$c^*\left(0, w', y, y'\right) = \min\left\{c\left(0, w', y, y'\right), g'\left(0, w', y, y'\right) + y\right\}. \tag{3}$$

This says that she cannot eat more than her labor income plus what he gives to her when she is broke, but she can announce plans to do so. Otherwise, realized consumption equals the announced strategy $c(0, w', y, y')$ because she faces no constraint. Strategically, this has the following consequence: fixing her strategy c, he knows that increasing

[7]Technically, we have to assume that these policy functions are continuously differentiable on the interior of the state space so as to ensure that value functions are twice differentiable and that the derivatives in the Hamilton–Jacobi–Bellman equation (7) exist in the conventional sense. Computationally, we solve the model by discrete approximation techniques that are robust to nondifferentiabilities, so that differentiability is not a practical issue for our results; see the discussion in the Computational Appendix.

his transfers will lead to higher realized consumption up to c and go into savings from then on. We define realized consumption $c'^{*}(w, 0, y, y')$ for him in the same manner.

When the other player's strategy is Markovian, the best-response problem of each player is a dynamic-programming problem and best responses are also Markovian. Let $x_t \equiv (w_t, w'_t, y_t, y'_t)$ denote the state vector, and let $v(x)$ and $v'(x)$ be the value functions for her and him. We will now heuristically derive the Hamilton–Jacobi–Bellman equation, which will provide the intuition for why working in continuous time is advantageous here.

Using Bellman's principle, we separate her problem into a trade-off between today (an interval of length Δt) and a continuation value tomorrow,

$$\begin{aligned} v(x_t) &= \max_{c,g\geq 0}\bigl\{u(c)\Delta t + \alpha u\bigl(c'(x_t)\bigr)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t v(x_{t+\Delta t})\bigr\} \\ \text{s.t.} \quad & \underbrace{w_{t+\Delta t} - w_t}_{\equiv \Delta w_t} = \underbrace{\bigl[rw_t + y_t + g'(x_t) - c - g\bigr]}_{\equiv \dot{w}_t}\Delta t + \sigma w_t \Delta B_t, \\ & \underbrace{w'_{t+\Delta t} - w'_t}_{\equiv \Delta w'_t} = \underbrace{\bigl[rw'_t + y'_t + g_t - c'(x_t) - g'(x_t)\bigr]}_{\equiv \dot{w}'_t}\Delta t + \sigma w'_t \Delta B'_t, \end{aligned} \tag{4}$$

where we take the continuation value v and his strategy $\{c', g'\}$ as given. $\Delta B_t = B_{t+\Delta t} - B_t$ and $\Delta B'_t = B'_{t+\Delta t} - B'_t$ are the shocks, that is, the increments of Brownian motion over Δt. We now take a second-order Taylor expansion of the continuation value around w_t and w'_t, fixing incomes y_t and y'_t at today's values for now,

$$\begin{aligned} & v\bigl(w_{t+\Delta t}, w'_{t+\Delta t}, y_t, y'_t\bigr) \\ &\quad \simeq v(x_t) + v_w(x_t)\Delta w_t + v_{w'}(x_t)\Delta w'_t \\ &\qquad + \frac{1}{2}v_{ww}(x_t)(\Delta w_t)^2 + \frac{1}{2}v_{w'w'}(x_t)\bigl(\Delta w'_t\bigr)^2 + v_{ww'}(x_t)\Delta w_t \Delta w'_t, \end{aligned}$$

where the subscripts on v denote partial derivatives. We now want to determine her expected continuation value at t. First, note that $\mathbb{E}_t[\Delta w_t] = \dot{w}_t\Delta t$ since the shocks have mean zero, that is, $\mathbb{E}_t[\Delta B_t] = 0$. By Ito calculus, we also have

$$\begin{aligned} \mathbb{E}_t\bigl[(\Delta w_t)^2\bigr] &\simeq \sigma^2 w_t^2 \mathbb{E}_t\bigl[(\Delta B_t)^2\bigr] = \sigma^2 w_t^2 \Delta t, \\ \mathbb{E}_t\bigl[\Delta w_t \Delta w'_t\bigr] &\simeq \sigma^2 w_t w'_t \mathbb{E}_t\bigl[\Delta B_t \Delta B'_t\bigr] = 0, \end{aligned}$$

where the last step follows since the two Brownian motions are uncorrelated. All other terms vanish since they are of order lower than Δt. We thus have

$$\begin{aligned} \mathbb{E}_t\bigl[v\bigl(w_{t+\Delta t}, w'_{t+\Delta t}, y_t, y'_t\bigr)\bigr] &\simeq v(x_t) + v_w(x_t)\dot{w}_t\Delta t + v_{w'}(x_t)\dot{w}'_t\Delta t \\ &\quad + v_{ww}(x_t)\frac{\sigma^2}{2}w_t^2\Delta t + v_{w'w'}(x_t)\frac{\sigma^2}{2}w_t'^2\Delta t. \end{aligned} \tag{5}$$

At this step, we see that the effects of the Brownian shocks on the continuation value are completely summarized by the second derivative of the value function. If she is locally risk-averse (i.e., $v_{ww}(x_t) < 0$), then these terms have a negative effect on the continuation value. The larger these shocks are, that is, the higher σ is, the more pronounced this negative effect is.

We now consider shocks to incomes y_t and y'_t. A change in income occurs with a probability $\xi\Delta t$ over a short horizon Δt for each player. Fixing wealth (w, w'), we can write, to a first order,

$$\begin{aligned} \mathbb{E}_t\big[v\big(w, w', y_{t+\Delta t}, y'_{t+\Delta t}\big)\big] \simeq{}& (1 - 2\xi\Delta t)v\big(w, w', y_t, y'_t\big) \\ &+ \xi\Delta t v\big(w, w', \tilde{y}_t, y'_t\big) + \xi\Delta t v\big(w, w', y_t, \tilde{y}'_t\big), \end{aligned} \tag{6}$$

where tildes denote the income state a player is currently not in (i.e., $\tilde{y}_t = y_l$ if $y_t = y_h$, etc.). Now, using Equations (5) and (6) in Bellman's principle (4), and approximating $e^{-\rho\Delta t} \simeq 1 - \rho\Delta t$, we obtain

$$\begin{aligned} v(x_t) \simeq{}& [1 - \rho\Delta t - 2\xi\Delta t]v(x_t) + \xi v\big(w_t, w'_t, \tilde{y}_t, y'_t\big)\Delta t + \xi v\big(w_t, w'_t, y_t, \tilde{y}'_t\big)\Delta t \\ &+ \max_{c,g\geq 0}\big\{u(c)\Delta t + \alpha u\big(c'(x_t)\big)\Delta t + \dot{w}_t v_w(x_t)\Delta t + \dot{w}'_t v_{w'}(x_t)\Delta t\big\} \\ &+ \frac{\sigma^2}{2} w_t^2 v_{ww}(x_t)\Delta t + \frac{\sigma^2}{2} w_t'^2 v_{w'w'}(x_t)\Delta t. \end{aligned}$$

At this step, it is important to note that we were able to discard several terms of order lower than Δt. First, and most importantly, instantaneous interactions between agents' decisions are of second order (i.e., $\dot{w}_t\Delta t\dot{w}'_t\Delta t v_{ww'}$). This simplifies our analysis of strategic interaction over a short horizon dramatically; more on this below. Second, interactions between income uncertainty and the instantaneous consumption decision are also of second order (i.e., $\xi\Delta t\dot{w}_t\Delta t V_w$). We can thus determine the optimal consumption decision solely from the marginal value of savings in the current income state, which is a key simplification that arises in continuous time.

Finally, divide by Δt and rearrange to obtain the Hamilton–Jacobi–Bellman equation (HJB), a partial differential equation (PDE) that her value function and its derivatives have to satisfy:

$$\begin{aligned} \rho v ={}& \max_{c,g\geq 0}\big\{u(c) + \alpha u\big(c'\big) + \dot{w}' v_{w'} + \dot{w} v_w\big\} \\ &+ \xi\big[v(\cdot, \tilde{y}) - v(\cdot, y)\big] + \xi\big[v\big(\cdot, \tilde{y}'\big) - v\big(\cdot, y'\big)\big] + \frac{\sigma^2}{2}\big(w^2 v_{ww} + w'^2 v_{w'w'}\big). \end{aligned} \tag{7}$$

We suppress the dependence of the function v on x for better readability. In the case that he (or she) is broke, we have to replace c' (or c) within the max operator by the realized-consumption function in (3). When she is broke, there is also the constraint $c + g \leq y$ on the controls. His problem is characterized by a mirror-symmetric HJB. Throughout the paper, we will only state her equations as long as the counterpart for him is obvious. We discuss the interpretation of the HJB as well as the constrained case in the following section.

We now have everything in place to define a recursive equilibrium.

Definition 1. A *Markov-perfect equilibrium (MPE)* is a collection of functions $\{v(\cdot), c(\cdot), g(\cdot)\}$ for her and $\{v'(\cdot), c'(\cdot), g'(\cdot)\}$ for him such that

1. $\{v(\cdot), c(\cdot), g(\cdot)\}$ solves her problem given $\{c'(\cdot), g'(\cdot)\}$, that is, they solve (7)
2. $\{v'(\cdot), c'(\cdot), g'(\cdot)\}$ solves his problem given $\{c(\cdot), g(\cdot)\}$.

Since players' strategies are required to be optimal for all points in the state space, players have to be best responding at any node of the game tree, even the ones off the equilibrium path. As is well known, Markov perfection thus implies subgame perfection.

3. Understanding players' incentives

3.1 *Best responses: HJBs*

We now return to her HJB (7) so as to shed light on her optimal choices in response to his consumption and transfer strategy. We will see that obtaining her best responses is akin to solving a standard consumption–savings problem.

We first focus on the case where both players are unconstrained. For convenience we reproduce her HJB, additionally writing out her and his savings policies, $\dot{w}$ and $\dot{w}'$, and group terms according to whether they can be controlled or not:

$$
\begin{aligned}
\rho v = {} & \underbrace{\alpha u(c') + (rw' + y' - g' - c')v_{w'}}_{\text{his decisions}} + (rw + y + g')v_w \\
& + \underbrace{\xi\big[v(\cdot, \tilde{y}) - v(\cdot, y)\big]}_{\text{shock to her } y} + \underbrace{\xi\big[v(\cdot, \tilde{y}') - v(\cdot, y')\big]}_{\text{shock to his } y'} + \underbrace{\frac{\sigma^2}{2}\big(w^2 v_{ww} + w'^2 v_{w'w'}\big)}_{\text{shocks to wealth}} \\
& + \max_{g\geq 0}\Big\{g\big[\underbrace{v_{w'} - v_w}_{\equiv\mu:\text{ transfer motive}}\big]\Big\} + \max_{c\geq 0}\Big\{\underbrace{u(c) - cv_w}_{\text{consumption–savings trade-off}}\Big\}.
\end{aligned} \tag{8}
$$

The key simplification of continuous time with respect to discrete time is that the players' four decisions do not contemporaneously interact with each other. The max operator for the transfer decision can be separated from the one for the consumption decision, and the other player's decisions do not enter these max operators.

Of particular importance is the fact that his contemporaneous consumption decision c' does not affect her optimal choice c. In other words, her best-response function over an infinitesimal amount of time is a constant. This simplification occurs for two reasons: first, the instantaneous payoff $u(c) + \alpha u(c')$ is separable in c and c'; second, the influence of his contemporaneous actions on her marginal value of saving, v_k, is small; recall, that in the derivation of the HJB, the term $\dot{w}'_t \Delta t \dot{w}_t \Delta t v_{ww'}$ vanishes. Economically speaking, for an agent who reconsiders her savings decisions on a daily basis, it is enough to keep an eye on the other's bank account to be sufficiently informed.

The first-order condition (FOC) for consumption is given by

$$
u_c(c) = v_w,
$$

which says that the marginal utility of current consumption equals the marginal value of saving in the optimum. At first glance, it seems striking that his current decisions should not matter to her. But note that this is only true for the decisions taken in the same instant of time. In general, his decisions do matter for her, which will become evident from her Euler equation. For now, bear in mind that the effects of his future decisions on her are all contained in the partial derivative v_w, which encodes the savings incentives stemming from the continuation of the game.

From (8), we see that the maximization problem with respect to the transfer g is linear or of the "bang–bang" type. If the term $\mu \equiv (v_{w'} - v_w)$, which we will refer to as the *transfer motive*, is negative, then transfers are set to zero—after all, she cannot force him to give transfers to her. If $\mu = 0$, then any transfer flow is consistent with optimality. Should the transfer motive be positive, however, then the agent wants to choose g as large as possible.

3.2 *Savings incentives: The Euler equation*

Just as in standard settings, most of the insights into the consumption–savings trade-off come from the Euler equation. To obtain it, take the derivative of her HJB (8) with respect to w, use the FOC for consumption, and rearrange:

$$\begin{aligned}&\underbrace{\mathcal{A}u_c(c)}_{\text{expected growth of } u_c}\\&\quad=\underbrace{(\rho - r)u_c(c)}_{\text{standard}} + \underbrace{\big[v_{w'} - \alpha u_c\big(c'\big)\big]c'_w}_{\text{altruistic-strategic distortion}} + \underbrace{\big[u_c(c) - v_{w'}\big]g'_w}_{\text{transfer-induced incentives}}.\end{aligned} \tag{9}$$

Here, c'_w and g'_w denote the partial derivatives of his policy functions with respect to w. The operator $\mathcal{A}$ (the *infinitesimal generator*) is defined for any twice-differentiable function $f(x)$ of the state x as the "expected time derivative":

$$\begin{aligned}\mathcal{A}f(x_t) &\equiv \lim_{\Delta t\to 0} \frac{1}{\Delta t}\mathbb{E}_t\big[f(x_{t+\Delta t}) - f(x_t)\big]\\&= f_w \dot{w}_t + f_{w'}\dot{w}_t + \frac{\sigma^2}{2}\big(w_t^2 f_{ww} + w_t'^2 f_{w'w'}\big)\\&\quad + \xi\big[f(\cdot, \tilde{y}) - f(\cdot, y)\big] + \xi\big[f\big(\cdot, \tilde{y}'\big) - f\big(\cdot, y'\big)\big].\end{aligned}$$

Introducing the operator $\mathcal{A}$ makes the right-hand side of the Euler equation identical to the one obtained in a deterministic environment; see Barczyk and Kredler (2014) for an extensive discussion of the strategic interactions described by this equation in a deterministic setting.

We now compare Equation (9) to the efficient benchmark. Consider a family planner who maximizes a convex combination of v and v'. This planner faces a standard consumption–savings problem, intertemporal optimality, being characterized by the standard Euler equation for a single agent (see Appendix A.1 for a formal derivation):

$$\mathcal{A}u_c(c_{\text{eff}}) = (\rho - r)u_c(c_{\text{eff}}). \tag{10}$$

We see that (9) and (10) differ by two distorting terms, which we refer to as an altruistic-strategic distortion and transfer-induced incentives. We will later find that transfers do not flow on the interior of the state space in equilibrium; thus transfer-induced incentives are zero. The interpretation for the altruistic-strategic distortion in (9) is, however, key to understand our results. Suppose his consumption is increasing in her wealth, that is, $c'_w > 0$. On the one hand, increasing her wealth w gives an instantaneous benefit $\alpha u_c(c')c'_w$ to her, which enters with the same sign as the interest rate and thus constitutes an incentive to save. On the other hand, there is a cost $v_{w'}c'_w$ that enters with the same sign as the discount rate does. The rise in his consumption causes his wealth to decrease, which she dislikes and thus induces her to save less. In equilibrium, the latter effect will prevail, inducing agents to overconsume with respect to the social optimum.

Finally, it is instructive to consider the special cases of selfishness and perfect altruism. In the absence of altruism ($\alpha = \alpha' = 0$), we have $c'_w = g'_w = 0$, that is, his behavioral response to her decisions is zero. Then there are no altruistic-strategic distortions or transfer-induced incentives, and we are left with the standard Euler equation, which says that marginal utility grows at the efficient rate $(\rho - r)$. On the other hand, in the fully altruistic case ($\alpha = \alpha' = 1$), the agents' criteria are identical and thus $v'_{w'} = v'_w = v_w = u_c(c) = u_c(c')$. The behavioral responses c'_w and g'_w are now nonzero, but the brackets in (9) vanish since players are in perfect agreement and thus indifferent with respect to the distribution of wealth between them.

3.3 *Optimality when broke*

The characterization so far was confined to the case where both agents have positive levels of wealth. Contrary to discrete time, in continuous time, a saver is unconstrained for any positive level of wealth, which simplifies our analysis and our computational algorithm substantially. We now discuss the important case where one agent is broke. For the discussion of the case where *both* agents are broke, we refer the reader to Appendix A.2.

Consider her problem when he is broke (i.e., $w' = 0$) but she has positive assets ($w > 0$).[8] To simplify matters, we leave out all terms in agents' HJBs that are not influenced by agents' contemporaneous decisions and focus on the Hamiltonians:

$$\begin{aligned}\max_{c\geq 0, g\geq 0} H(c,g) = \max_{c\geq 0, g\geq 0}\big\{&u(c) + \alpha u\big(c'^{*}\big(c'_0, g\big)\big) + (rw + y - c - g)v_w \\ &+ \big(y' - c'^{*}\big(c'_0, g\big) + g\big)v_{w'}\big\},\end{aligned} \tag{11}$$

$$\max_{c'\geq 0} H'\big(c'\big) = \max_{c'\geq 0}\big\{u\big(c'\big) + \big(y' - c'^{*}\big(c', g\big) + g\big)v'_{w'}\big\}. \tag{12}$$

We now guess that $g' = 0$ and will later verify this. Recall that c' is his consumption strategy; realized consumption c'^{*} for him is obtained from g and c' as prescribed by Equation (3):

$$c'^{*}\big(c', g\big) = \begin{cases} c' & \text{if } y' + g \geq c', \\ y' + g & \text{if } y' + g < c'. \end{cases}$$

[8] The case $w' > 0$ and $w = 0$ is entirely symmetric.

We will now show that agents' optimal consumption strategies are given by the "unconstrained levels" (c_0, c_0'), which we implicitly define as

$$
\begin{aligned}
&u_c(c_0') = v_{w'}', \\
&u_c(c_0) = v_w.
\end{aligned}
\tag{13}
$$

Obviously, c_0 maximizes H and is thus the optimal strategy independent of c_0'. As for him, observe that the unconstrained maximum of H' is reached at c_0'. So setting c_0' is definitely optimal if it is feasible. Also, since H' is increasing in c' for $c' < c_0'$, announcing c_0' leads to the constrained optimal outcome $c'^* = y + g$ in the case that $c_0' > y + g$. This means that announcing c_0' is a dominant strategy for him, and in case that he is constrained, it always leads to the best outcome for him.[9]

To analyze her transfer decision, it will be convenient to introduce a variable g_{dict} that tells us what she would like to give to him (or take from him) if she could dictate his consumption in the current instant. We define

$$
\begin{aligned}
&g_{\text{dict}} \equiv \arg \max_{-\infty < \tilde{g} < \infty} H(c_0, \tilde{g}) \\
&\quad \Rightarrow \quad \alpha u_c\big(y' + g_{\text{dict}}\big) = u_c(c_0).
\end{aligned}
$$

For the case of constant elasticity of substitution (CES) utility, for example, we have $g_{\text{dict}} = \alpha^{1/\gamma} c_0 - y'$.

We will now study the properties of the Hamiltonian $H(c, g)$ in Equation (11). Since utility is separable in c and c', H is additive in its c and g terms. This greatly simplifies our analysis. We see that H is concave in c and that

$$
\frac{\partial H}{\partial c} = u_c(c) - v_w \begin{cases} \geq 0 & \text{if } c \leq c_0, \\ < 0 & \text{if } c > c_0. \end{cases}
$$

H is continuous in g, but there is a kink at the point where he starts saving the additional transfer instead of consuming it:

$$
\frac{\partial H}{\partial g} = \begin{cases} \alpha u_c\big(y' + g\big) - v_w & \text{if } g < c_0' - y', \\ v_{w'} - v_w & \text{if } g > c_0' - y'. \end{cases}
$$

The jump in the derivative reflects that her transfers go directly into his consumption until the satiation point c_0' is reached. On this lower part, $H(c, \cdot)$ is concave and may (or may not) reach a local maximum. This local maximum—if it exists—occurs at g_{dict}. From the satiation point on he starts saving, which is marginally valued by her at $v_{w'}$. We are guessing an equilibrium in which transfers are never optimal within the state space, so let us assume $v_{w'} < v_w$ for now. In this case, she never desires to raise g to the point where he saves additional transfers.

We will distinguish between two cases: in Case 1, he saves even when transfers are zero, that is, $c_0' < y$. In Case 2, he consumes the marginal transfer at $g = 0$, that is, $c_0' > y$.

[9] Barczyk and Kredler (2014) have an explicit formulation of a transfer and a (subsequent) consumption stage, and show that this is the unique equilibrium of the stage game.

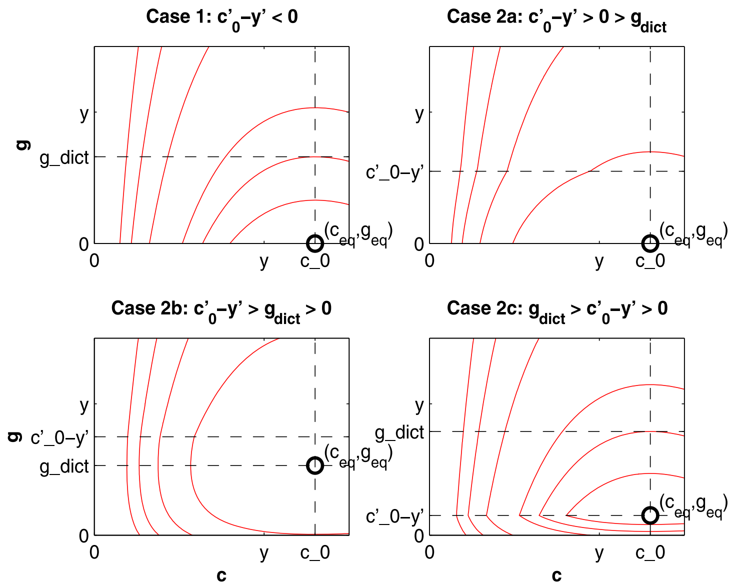

FIGURE 1. Transfer decision when $w > 0$.

We will denote her optimal transfer choice by g_{unc} (unc is for unconstrained, in contrast to the case where she is also constrained).

Figure 1 provides an illustration of these two cases. It plots level lines of her Hamiltonian $H(c, g)$ as curves and the optimal choice as a circle. The graph in the upper-left corner depicts Case 1. She sets $g_{\text{unc}} = 0$ since all transfer units would go into savings, but $v_{w'} < v_w$. We may say that her *transfer margin* (the marginal benefit of increasing transfers) is lower than the *savings* and the *consumption margins* (the marginal benefits of savings and consumption, respectively).

The remaining graphs refer to the Case 2, which is split up into further subcases. In Case 2a ($g_{\text{dict}} \le 0 < c_0' - y'$), she is already unwilling to give transfers to him at $g = 0$, that is, $\partial H/\partial g|_{g=0} \le 0$. Then the optimal transfer is zero since H is decreasing in g throughout and thus $g_{\text{unc}} = 0$. Again, her transfer margin is lower than the consumption and savings margins.

In Case 2b ($0 < g_{\text{dict}} < c_0' - y'$), there is an interior solution where he consumes the entire transfer and she sets the c and g margin equal, so she can implement her desired consumption for him. In this case, she equalizes the consumption, savings, and transfer margins, and we end up with her favored allocation over the next instant Δt.

Finally, Case 2c shows that another corner solution can occur where she increases transfers until reaching his satiation point where he would start saving the transfer, which she dislikes. In this case, she gives $c_0' - y'$ and we have $0 < c_0' - y' \le g_{\text{dict}}$ so that he is just indifferent between saving and consuming the marginal transfer unit. The transfer margin is now *lower* than the consumption and savings margins when considering a local *increase* in transfers at the optimum, and is *higher* than the consumption and savings margins when considering a marginal *decrease* in transfers at the optimum. Graphically, this shows ups as kinks in the level lines of the Hamiltonian.

Computationally, we later find that in equilibrium all cases apart from Case 2c occur. We can summarize all cases by defining her optimal transfer by the formula

$$g_{\text{unc}} = \max\{0; \min\{g_{\text{dict}}, c_0' - y'\}\}. \tag{14}$$

The maximizers for problem (11) are thus $(c, g)_{\text{eq}} = (c_0, g_{\text{unc}})$. His equilibrium consumption is given by using his policy rule for her optimal choice g_{unc},

$$c_{\text{eq}}'^{*} = \min\{c_0', y' + g_{\text{unc}}\} = y' + g_{\text{unc}}, \tag{15}$$

where the second equality follows because she never gives a transfer that would flow into his savings.

4. Results

4.1 *Algorithm*

Our algorithm uses the Markov-chain approximation method (see Kushner and Dupuis (2001)). The state space is discretized on a linear grid, and the law of motion for the state is approximated by a Markov chain that is locally restricted to adjacent grid points. If the local properties of the Markov chain are chosen such that they are in line with the true continuous-time process (in first and second moments), then it is known that the discretized solution converges to the continuous-time solution as the grid becomes finer (see again Kushner and Dupuis (2001)). The algorithm can also be seen as a traditional finite-difference PDE approximation scheme for the two HJBs. The Computational Appendix provides details of the algorithm as well as an introduction to the Markov-chain approximation method for continuous-time problems and its connection to finite-difference methods.

Our algorithm is closely related to value-function iteration in discrete time. Since the equilibrium-selection criterion is to pick equilibria that are the limit of finite games, the algorithm starts at a final time interval $[T - \Delta t, T]$ of short duration. The first step is to determine a reasonable consumption allocation over this final interval. It turns out that letting players play a static altruism game in transfers over this final interval leads to numerical instability since the value functions of this game have kinks (as is well known). A more stable approach is to choose a consumption allocation that is smoothly increasing in wealth. The precise form of this rule is not essential for the algorithm's results. We have found that all reasonable (i.e., smooth, monotone, and concave) specifications of final value functions at $T - \Delta t$ consistently yield the same equilibrium in the limit.

We then iterate backward, computing optimal policies in the following manner. On grid points where he has positive assets, her transfers are set to zero and her consumption rate is determined from her marginal value of assets. As pointed out before, it is very easy to find the equilibrium of the "stage game" when both players are unconstrained since best responses are constant in our continuous-time setting. On grid points where one or both players are broke, the algorithm follows the results from Section 3.3 and Appendix A.2. We proceed with the algorithm backward in time until value functions converge. Due to the simplicity of the optimal policy rules in the stage games, the algorithm is very fast and computation time is not an issue.

Once value functions have converged, we have to check that the transfer motive is indeed negative for both players throughout the state space and over time, which justifies that we set transfers to nonbroke recipients to zero in the algorithm. If this is the case, then the transfers-when-constrained equilibrium is the unique equilibrium according to our selection criterion. For the specification $u(c) = c^{1-\gamma}$ and $\alpha = \alpha'$, we find that for reasonable parameter configurations $\{\rho, \gamma, \alpha, y, y', \xi\}$, there always exists σ large enough such that this type of equilibrium exists. The values for σ that are required are 2–3% yearly. We further discuss the existence and uniqueness of the equilibrium and the role the shocks play in Section 4.3.

4.2 *Equilibrium properties*

The equilibrium can be understood as being made up of three distinct regions. We refer to them as *transfer regions, overconsumption* (OC) *regions*, and *self-sufficient* (SS) regions. Before going into more detail we describe the key features of these regions. The reader may glance at Figures 2 and 3 so as to follow the discussion in this part better.

The transfer regions are defined by a positive transfer flow. Transfers are given by relatively wealth- and (labor-)income-rich donors to poorer recipients. They only flow once the recipient has exhausted his assets and is borrowing-constrained. In this way, the donor can control the consumption behavior of the recipient and temporarily takes on the role of a family dictator, that is, the donor implements the allocation that is in his best interest for the time that transfers flow. Positive labor income shocks to the recipient (and negative labor income shocks to the donor) cause transfers to decline or stop entirely.

The OC regions border the transfer regions with both players' wealth being positive. In these regions, one agent is poor relative to the other and so the wealth distribution is relatively unequal. In anticipation of transfers, the poorer player spends down her wealth and the economy heads toward the transfer region. A defining feature of equilibrium policies in these regions is moral hazard. The poor agent behaves recklessly by overconsuming, counting on the benevolence of the altruistic donor. Meanwhile, the richer agent is also overconsuming. The resulting inefficiencies are similar to the tragedy of the commons. Both agents ultimately use a common resource (the donor's assets), which leads both agents to overconsume long before transfers actually occur. We call this result the *dynamic Samaritan's dilemma*. Finally, the recipient's consumption path exhibits a downward discontinuity upon entering the transfer region.

The SS region comprises the rest of the state space. Here, the wealth distribution is relatively balanced. The possibility of future transfers is remote, and consumption policies resemble those that agents would choose in the absence of a second altruistic agent. The allocation is close to efficient in this region.

4.2.1 *Numerical example* We now study a numerical example so as to illustrate the central features of the equilibrium. The numerical example is without loss of generality in the sense that all transfer-when-constrained equilibria display the same qualitative features as long as both agents are at least imperfectly altruistic. Quantitatively,

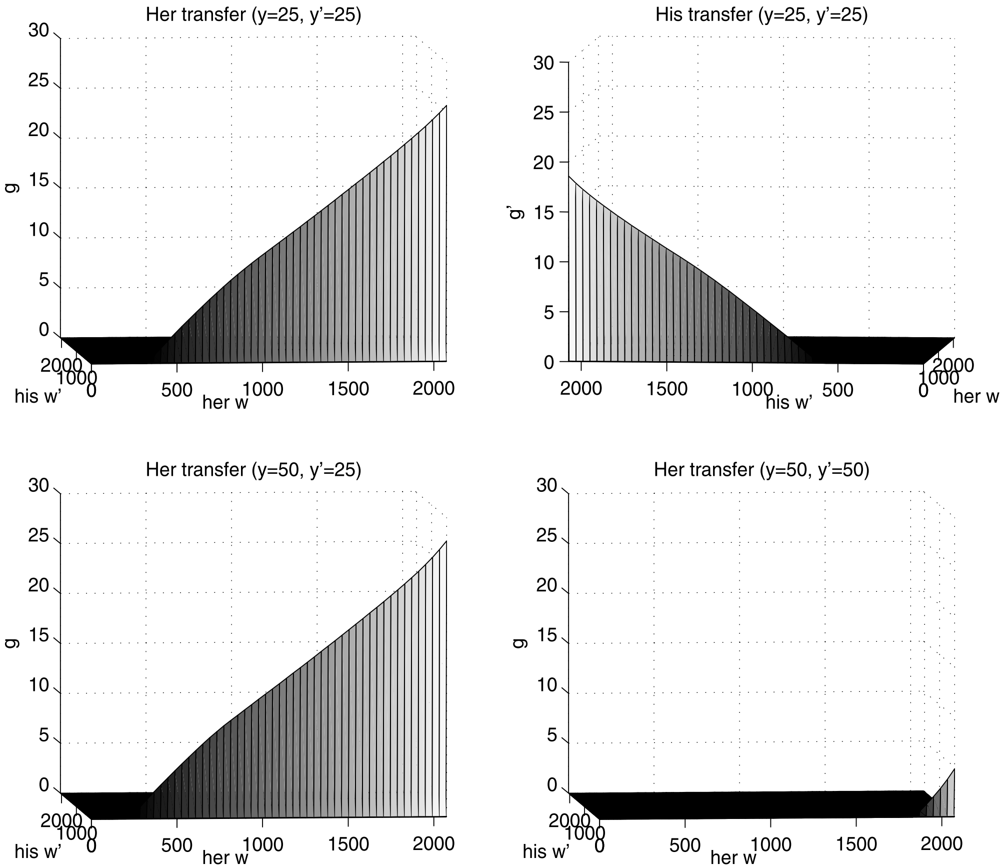

Figure 2. Transfer policies. Not shown are her transfers for $\{y = 25, y' = 50\}$, his transfers for $y = 50$, and $\{y = 25, y' = 50\}$. His transfers are zero throughout the grid when $y = 50$ (but would become positive for high wealth levels w' if a larger grid was chosen).

of course, the salience of these features varies depending on the particular parameter values. Appendix A.3 discusses changes to the altruism parameters, in particular, the special cases of selfishness, perfect altruism, and one-sided altruism, and studies the robustness of our results with respect to the other model parameters.

Table 1 provides a summary of the parameter values for the baseline example. We choose standard values for r and γ. Our value for ρ is slightly lower than in one-agent models, since the model generates too many binding constraints otherwise. Both agents face the same process for labor income, but she is more altruistic than he is, that is, $\alpha > \alpha'$.

4.2.2 *Transfer policies* Players' equilibrium transfer policies are shown in Figure 2.

Transfers only flow when the recipient has no wealth left. Once the donor's wealth becomes too low, transfers stop altogether in this example. It is, in general, possible, however, that transfer regions extend all the way to the origin, that is, that an agent

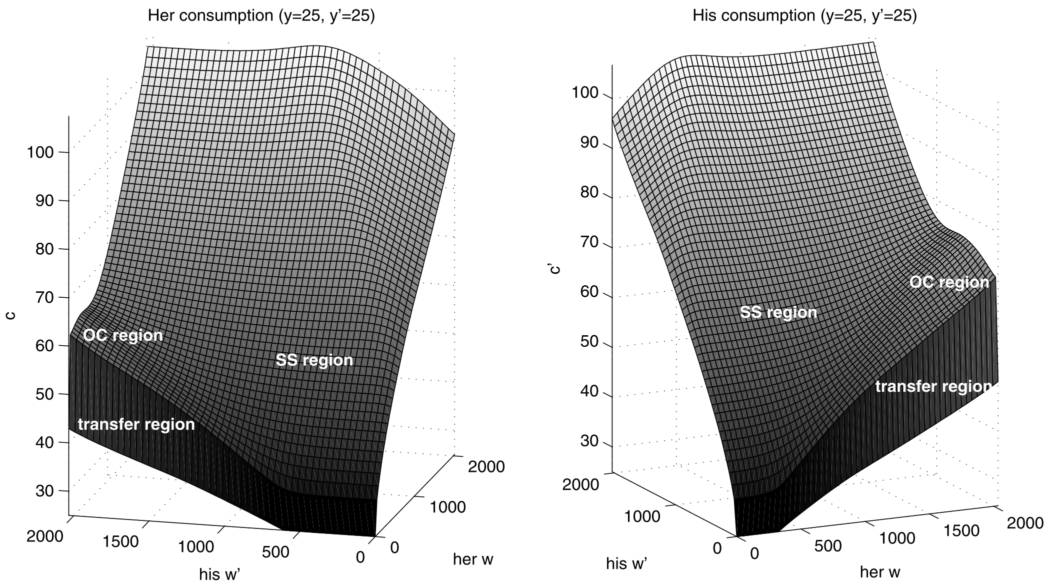

FIGURE 3. Consumption policies. This figure depicts realized consumption c^*. Not shown are cases where either player's labor income is high.

TABLE 1. Parameters in the numerical example.

Parameter	Value	Parameter	Value
α	0.3	y_h, y'_h	50
α'	0.2	y_l, y'_l	25
γ	2	ξ	10%
ρ	3.4%	ξ'	10%
r	3%	σ	3%

with zero wealth gives transfers. This situation occurs when labor-income differences are large and altruism of the donor is high.

Furthermore, transfers are increasing in the donor's level of wealth (all panels), increasing in the donor's labor income (compare the two left panels), and decreasing in the recipient's labor income (compare the two lower panels). As mentioned above, all these features are in line with the empirical evidence on inter vivos transfers.

Considering the transfer policies, it is noteworthy that transfers react very strongly to an increase (decrease) in the recipient's income, but much less to an increase (decrease) in the donor's income. We see that a change in his income, for example, an increase from $y' = 25$ to $y' = 50$ shown in the lower two panels, leads to a sizeable reduction in transfers of roughly 25 units, leaving the recipient's consumption almost stable. In contrast, if she receives an income shock, for example, an increase from $y = 25$ to $y = 50$ seen in the two left panels, transfers react comparatively weakly, again keeping the recipient's consumption stable. The reason for the different reaction to the two income shocks is that she is smoothing both her and his consumption using her stock of wealth as a buffer.

Finally, note that since she is more altruistic than he is ($\alpha > \alpha'$), her transfers are more generous than his; compare the two top panels.

4.2.3 *Consumption, savings, and equilibrium dynamics* Players' (realized) consumption in equilibrium is depicted in Figure 3. We only show consumption for the case in which both players have low labor income since policies for the high labor income state are qualitatively very similar (although they are higher in levels, especially in regions where wealth is low). Unsurprisingly, consumption is increasing in both own and the other player's wealth.

Consider first the OC region in her consumption function. This OC region covers an area where she is asset-poor and he is wealthy, and so she expects to obtain transfers in the foreseeable future. This explains why in the OC region her consumption depends more on his assets than on her own. A striking feature is that her consumption jumps downward when she goes from positive to zero wealth. The region in which this jump takes place corresponds to the region in which he provides transfers. Figure 2 shows that in the case of $\{y = 25, y' = 25\}$, he provides transfers when $w' > 600$; essentially, once the economy reaches the transfer region, he restricts her consumption to what he deems desirable. We discuss the optimality of the consumption discontinuity further in Section 4.2.6. This downward jump does not occur when his wealth is low and he gives no transfers; in this case, we observe standard consumption–savings behavior. The donor's consumption policy, however, evolves continuously when she moves from being solvent to broke; after all, he is always in control of his consumption and thus we see the familiar consumption smoothing. From his consumption policy, we see that the situation is reversed when she is wealthy and he is poor and receives transfers.

Within an OC region, the economy is moving toward a transfer region. Figure 4 visualizes the dynamics by plotting the economy's law of motion. The heavy solid arrows depict the law of motion, or drift, $(\dot{w}, \dot{w}')$ for a given state (w, w') in equilibrium. To put our results into perspective, we contrast the law of motion of the imperfect-altruism economy with the law of motion of a no-altruism economy, that is, an economy in which $\alpha = \alpha' = 0$ but where all other parameters are held constant (the dashed arrows). We see that in his OC region (the lower-right corner of all panels), the altruistic economy heads rapidly toward the region in which he obtains transfers. In the self-sufficient/no-altruism economy, however, he always shows typical precautionary-savings behavior: he saves when his labor income is high, that is, the dashed arrows point upward in the two right panels where his income is $y' = 50$, and dis-saves when his labor income is low, that is, the dashed arrows point downward in the two left panels where his income is $y' = 25$. As for *her* consumption policy in this region, it is worthwhile to point out that she often consumes less (or saves more) than in the self-sufficient/no-altruism benchmark as her wealth decreases since she foresees that she will have to provide for him eventually.[10]

The SS region is the region in which the wealth distribution is relatively balanced. In this part of the state space, players' consumption is determined mainly by their own

[10]However, her consumption path is still inefficiently front-loaded relative to the social optimum, as Section 4.2.7 will show.

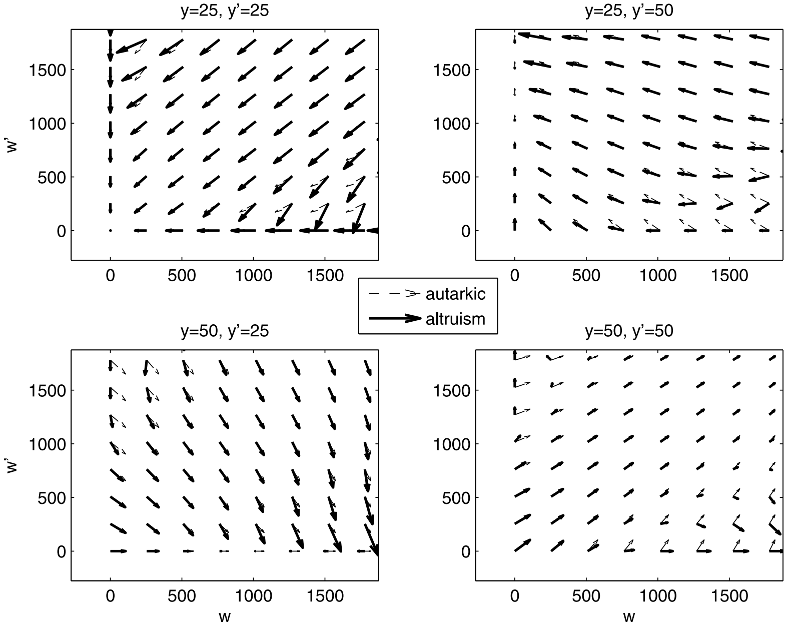

FIGURE 4. Law of motion/drift of asset position. The heavy solid arrows denote the drift in the altruism economy (baseline model); the dashed arrows denote the drift in the no-altruism economy ($\alpha = \alpha' = 0$; other parameters are as in the baseline model).

assets: her consumption function is increasing in her assets w and flat in his assets w' (and the opposite for him). Figure 4 shows that players' policies, and thus the law of motion of the economy, are practically the same as in the self-sufficient economy. This is because it is unlikely that the economy ends up in a region where one player becomes dependent on the other.

Take a closer look at her and his consumption functions in the upper-right corner of the state space where she has $w = 1500$ and he has $w' = 500$. It is very interesting to observe what occurs on the seams between the SS and the OC regions. On the boundary between the SS and the OC regions, we see that *both* players' consumption functions have upward bumps as the economy leaves the SS region: for both him and her, consumption is higher at this bump than would be expected when linearly extrapolating from within the SS region. The reason for this is that both players have incentives to overconsume as they enter the OC region—the dynamic Samaritan's dilemma. For him, this effect indeed leads to the consumption function being locally decreasing in his own wealth. This is equivalent to the value function being convex, meaning that he is locally risk-loving. We will return to this feature of the equilibrium in Section 4.2.8.

From the phase diagram in Figure 4, we see that once the economy crosses this seam it becomes increasingly unlikely that the economy returns to the SS region. As a result, the future donor curbs her consumption in anticipation of having to provide transfers. The future recipient, on the other hand, does not fully internalize the effects of his consumption behavior on her resources and overconsumes. As a final note, Figure 4 reveals that the OC region in the lower-right corner is larger than in the upper-right corner. This is because her altruism is higher than his, which induces him to rely on her in a wider range of circumstances.

4.2.4 *Ergodic distribution* Figure 5 informs us about the ergodic distribution of wealth. The contour lines in the upper-right quadrant show the level lines of the ergodic density over (w, w'). Note that due to the wealth shocks, there is always a chance that both players wealth grows large, so the support of the ergodic distribution is the entire $\mathbb{R}^2_+$. Since both players are saving when facing high labor income, the economy always comes back into the region where both players have positive wealth. We see that she usually holds higher wealth than he does, since he relies more on her transfers than she does on his.

Indeed, in the example, we find that she gives transfers to him 10.0% of the time, whereas he gives transfers to her only 2.9% of the time. The curve in the lower-right quadrant gives the mass distribution over states $\{(w, w') : w > 0, w' = 0\}$, which has a total mass of 18.7% under the ergodic distribution. So 8.7% of the time he is broke but does not receive transfers from her. The upper-left quadrant shows the distribution over cases where only she is broke, which is the case 9.6% of the time. Finally, there is a mass point at the origin where both players are broke, in which the economy spends 0.7%

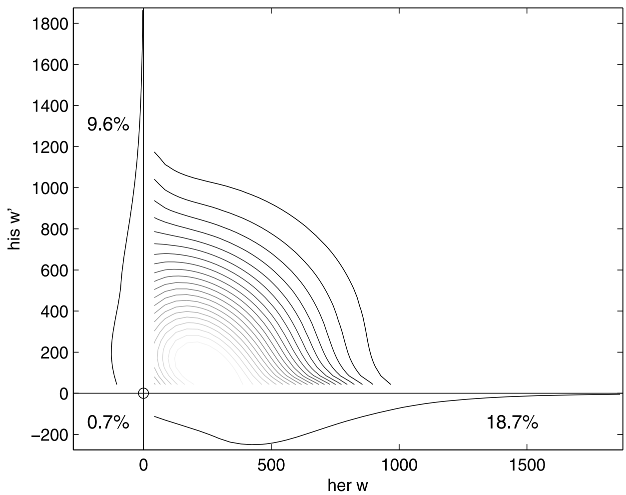

FIGURE 5. Contours of the ergodic distribution. Income dimensions are integrated out. The upper-right quadrant shows contour lines of the density for situations where both players have positive wealth. The upper-left quadrant depicts the mass region where she is broke but he has positive assets ($w = 0, w' > 0$) as a univariate density over w' (total mass: 9.6%). The lower-right quadrant shows the distribution for the case ($w > 0, w' = 0$), total mass: 18.7%. The lower-left quadrant gives the size of the mass point where both are broke simultaneously ($w = w' = 0$; 0.7% of time). In total, she is broke 10.3% of the time and he is broke 19.4% of the time. She gives transfers 10.0% of the time and he gives transfers 2.9% of the time (not shown in figure).

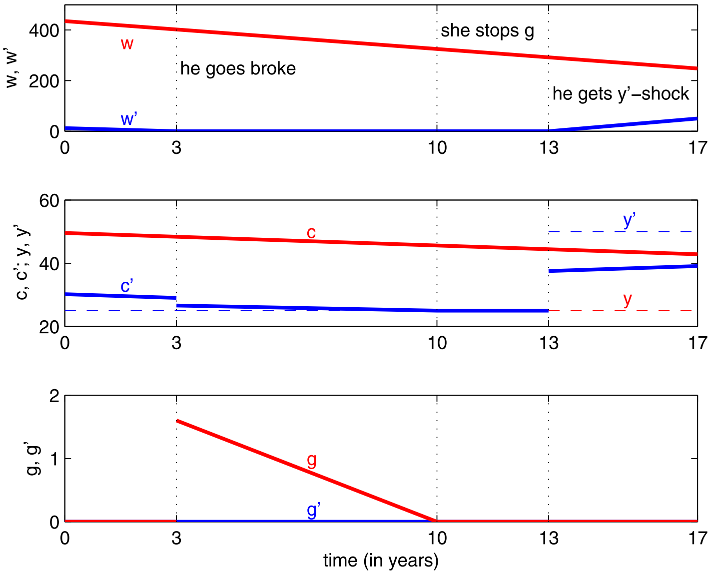

FIGURE 6. A history of wealth, consumption, and transfers. Both players start with $y_0 = y'_0 = 25$, the only switch in income occurring for him at $t = 13$. Wealth shocks are set to zero for illustrative purposes.

of the time. In the corresponding no-altruism economy, agents are only broke 6.6% of the time. Thus, our model may provide a way to address one of the well known shortcomings of the Bewley model, which is that it does not generate enough wealth-poor households.

4.2.5 *A history* We can gain a better appreciation of the equilibrium dynamics by considering a particular history for players' assets, consumption, and transfers. Figure 6 displays such a history. The graph on the top displays her and his wealth trajectories. She starts out with high assets and he is relatively wealth-poor at $t = 0$ (which corresponds to starting the economy in the lower-right corner of the plots in Figure 4). As for labor income, we let both agents start with the low realization and make him switch to the high income after 13 years, as we see in the second panel. Until the third year, he consumes his wealth down, at which point he obtains transfers as shown in the bottom panel. Notice that his consumption path jumps downward at the point when he obtains transfers, which corresponds to the discontinuity of his consumption function in Figure 3. She then provides him with transfers from year 3 to year 10. These transfers decrease over time (since she is spending down her wealth) and eventually stop. For 3 years, he then consumes only his labor income. In year 13, he obtains the high income realization and as a result begins to accumulate wealth, her transfers remaining zero.

4.2.6 *Characterizing incentives in OC region* We now return to the discontinuity in the recipient's consumption path when entering the transfer regime. We find that the size of

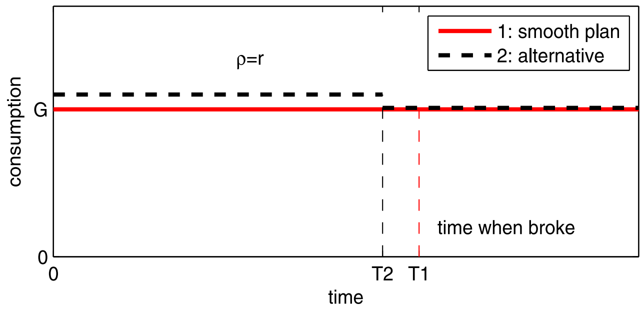

FIGURE 7. Means-tested benefit G. Would you prefer plan 1 or plan 2?

the jump is decreasing in both altruism parameters. The more agents take into account the effects of their behavior on the other, the smaller the inefficiency becomes.[11]

The intuition for why a jump in consumption can indeed be optimal in the presence of conditional transfers can already be gleaned from a simple example of means-tested benefits in the absence of altruism. Consider a consumer with wealth $w_0 > 0$ and no labor income. A government provides a means-tested benefit in the form of a flow payment G handed out conditional on the agent having zero wealth. For simplicity, assume that $\rho = r$, which implies that the optimal consumption path is constant while assets are positive, that is, $c_t = \bar{c}$ for all t for some constant $\bar{c}$. Given w_0, the agent will be able to consume $\bar{c}$ over an interval $t \in [0; T(\bar{c})]$, where $T(\bar{c}) \in (0; \infty]$ is the insolvency time implied by the consumption plan.

Consider the two consumption paths depicted in Figure 7. A smooth consumption path implies $\bar{c} = G$. As is obvious from the figure, any plan with $\bar{c} > G$ does better than this, so a smooth consumption plan cannot be optimal. In technical terms, the familiar theorems from control theory fail because the law of motion $w_t = rw_t + c_t + GI_{w_t=0}$ is a discontinuous function of the state at $w = 0$. In terms of marginal cost–benefit analysis, a means-tested benefit adds an additional cost of saving to the standard consumption–savings trade-off: in postponing bankruptcy, a saver diminishes the net present value of government transfers.

For an altruistic transfer recipient, the situation is similar. However, the additional disincentive to save is not quite as pronounced since an altruistic recipient takes the effects of her behavior on the donor into account. To see this, suppose she has a small level of wealth Δw left. Define Δt as the time it takes her to reach zero assets given a consumption rate c. Her problem, taking his policy c' and thus $\dot{w}'$ as given, may be written as

$$\max_{c \geq 0} \bigl\{ u(c)\Delta t + \alpha u\bigl(c'\bigr)\Delta t + e^{-\rho \Delta t} v\bigl(0, w' + \dot{w}'\Delta t; y, y'\bigr) \bigr\},$$

[11] Barczyk and Kredler (2014) derive a closed-form expression for the size of this jump in the same setting without flow labor income and with logarithmic preferences. They argue that this discontinuity is the equivalent to the Samaritan's dilemma in two-period models.

where

$$\Delta t = \frac{\Delta w}{c - y - r\Delta w}.$$

The first-order condition with respect to c is

$$\underbrace{u_c(c)\Delta t}_{\text{flow utility}} + \frac{\mathrm{d}\Delta t}{\mathrm{d}c}\big[\underbrace{u(c) + \alpha u(c') - \rho V}_{A} + \underbrace{\dot{w}' v_{w'}}_{B}\big] = 0.$$

An increase in c leads to additional flow utility, $u_c(c)\Delta t$, but exhausts her wealth earlier, that is, $\mathrm{d}\Delta t/\mathrm{d}c < 0$. Expression A says that exhausting wealth earlier replaces flow utility before going broke, $u(c) + \alpha u(c')$, with the flow value of being broke, ρV. The term A will usually be positive, since consumption just before being broke is higher than consumption when broke. So term A acts as an incentive to delay the point of going broke, that is, it is an incentive to save. Depending on the sign of $\dot{w}'$, expression B either provides an incentive or a disincentive to save. In our example history from before, he was dis-saving when giving transfers (i.e., $\dot{w}' < 0$). This would be an additional disincentive to save for her. Consuming more makes her face a richer donor upon entering the transfer regime, which increases the transfers she can expect.

The discontinuity in the recipient's consumption path is a stark deviation from efficiency. The poor agent behaves recklessly by overconsuming, counting on the benevolence of an altruistic donor. This is a form of moral hazard: if agents' consumption and transfer decisions were contractible, this inefficiency could be avoided. It turns out that in equilibrium there are also other sources of inefficiency, which we turn to next.

4.2.7 *Inefficiencies* We first introduce a measure for inefficiency of an allocation. An allocation is a contingent plan for consumption and transfers for both agents for $t \in [0, \infty)$. Whenever an inefficient allocation is played at some point $x = (w, w', y, y')$ in the state space, then there exists a continuum of efficient allocations indexed by Pareto weights η that is preferred by both players. The Pareto weights associated with these preferred allocations lie in a range $\eta \in [\underline{\eta}(x), \bar{\eta}(x)]$. Depending on η, the efficiency gains are shared differently: the $\underline{\eta}(x)$ allocation gives all gains to him, while the $\bar{\eta}(x)$ allocation gives all gains to her. So as to have a unique measure for the potential welfare gains, we will focus on the efficient allocation that provides the same gain to both players' welfare in the sense of consumption equivalent variation.

Formally, consider the following thought experiment. At a given point x in the state space, offer the efficient allocation with Pareto weight η to both agents. Compute the percentage increase $\gamma(x, \eta)$ in consumption (for all future t, for all states of the world, and for both players) that she requires to be indifferent between the offered η allocation and the equilibrium allocation. Equivalently, compute the percentage increase $\gamma'(x, \eta)$ that he would require. We can then compute γ and γ' for all efficient allocations $\eta \in [0, 1]$, as is illustrated in Figure 8 for one particular x. In this example, she is sufficiently well off under the equilibrium allocation to reject any efficient allocation that assigns weight lower than $\underline{\eta}(x) = 0.5$ to her, whereas he would accept any allocation with $\eta <$

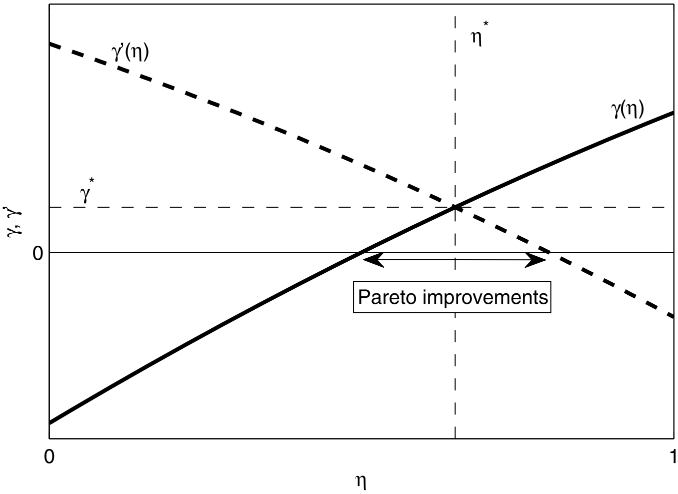

FIGURE 8. Potential welfare gains for a given state $x = (w, w'; y, y')$.

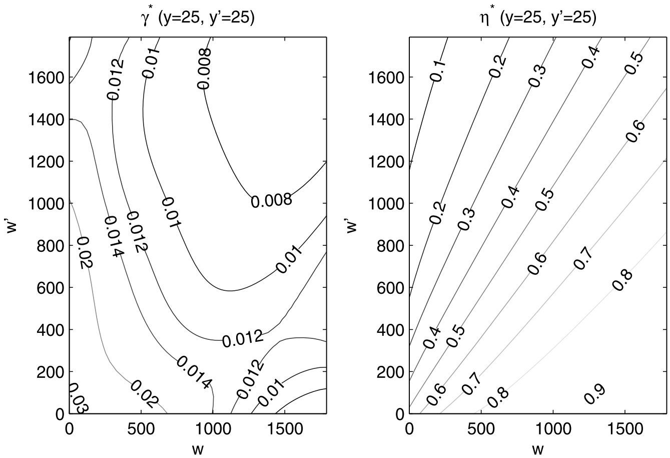

FIGURE 9. Quantifying distortions: consumption equivalent variation. Not shown are income states where either player has high income. The average inefficiency (γ^*) using ergodic weights is 2.0%.

$\bar{\eta}(x) = 0.8$. There is an interval $[\underline{\eta}(x), \bar{\eta}(x)]$ of allocations that make both players better off. This interval must contain at least one point by the definition of Pareto efficiency.

The efficient allocation that gives equal gains to both agents is associated with the value η^* that solves $\gamma(x, \eta^*) = \gamma'(x, \eta^*)$. Obviously, the intersection of γ and γ' that identifies this value must be unique. The common welfare gain associated with this allocation is $\gamma^*(x) = \gamma(x, \eta^*) = \gamma'(x, \eta^*)$ and is our measure for the inefficiency of the equilibrium allocation at x. Using this procedure, we now obtain the potential welfare gains for all points in the state space.

Figure 9 shows contour lines of γ^* and η^* for our numerical example.

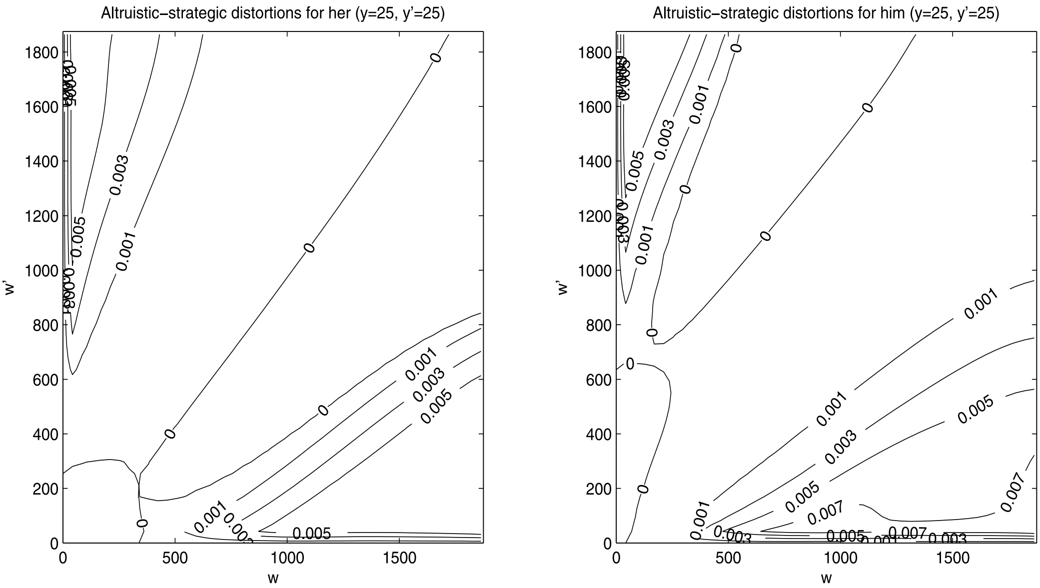

FIGURE 10. Altruistic-strategic distortions in Euler equation. The figure depicts the altruistic-strategic distortion in Euler equation (9) as a fraction of marginal utility (i.e., $c'_w[v'_w - \alpha u_c(c')]/u_c(c')$). This number gives us the growth rate of marginal utility in excess of the efficient rate $\rho - r = 0.4\%$. The average distortions using ergodic weights is 0.03% for her and 0.06% for him.

In the SS region, we see that the allocation is close to efficient: recall that players' policies are close to the SS policies, which satisfy the planner's Euler equation (10). Similarly, Figure 10 plots the altruistic-strategic distortions for her and him from the Euler equation (9) and shows that the Euler equations are essentially undistorted in the SS region.[12]

For the OC region, Figure 10 shows that distortions to the Euler equations (EE) are large and positive for the poorer player: his growth rate of marginal utility is up to 0.7% higher than the efficient rate $\rho - r = 0.4\%$ in the lower-right corner. There are also positive distortions to the richer player's EE, but they are somewhat weaker. These distortions entail what we call the dynamic Samaritan's dilemma: inefficiencies feed back in time to long before transfers start to flow, and both players are overconsuming.[13] The intuition behind this type of inefficiency is akin to the tragedy of the commons. Both agents ultimately consume out of a common resource—the donor's assets. Agents take into account the adverse consequences of their behavior on the other person, but fail to do so completely because their altruism is imperfect and thus they engage in a race to the bottom. In fact, we find that the less altruistic is a player, the stronger are the distortions to his Euler equation.

[12]Note that since transfers are zero inside the state space, transfer-induced incentives are zero and we can focus solely on the altruistic-strategic distortion when analyzing distortions to the EE.

[13]In the two-period models studied before in the literature, only the recipient's savings decision is distorted and, by the two-period assumption, distortions are only present immediately before transfers flow.

Figure 10 shows that agents' Euler equations are undistorted in the transfer regions, the distortions sharply dropping with respect to the OC region. This is because the donor effectively acts as a family dictator inside the transfer region. Figure 9 confirms this; it shows that the rich player obtains a value very close to his/her preferred allocation (the implied Pareto weights are close to 0 or 1). Figure 9 further shows that when the recipient has little wealth left, the allocation is still almost efficient. The reason for this is that the OC inefficiency is short-lived at this point: there is only a little time left until the economy enters a close-to-efficient regime. However, the criterion $\gamma^*(x)$ is about the consumption allocation for the entire future. When moving farther into the OC region, more time is spent in a distorted regime and γ^* increases.

Finally, due to the presence of labor-income risk, a type of inefficiency known from (no-altruism) Bewley models is also present in our setting. In Figure 9, we see that the equilibrium allocation is inefficient in the lower-left corner, where she is broke and he does not provide transfers to her. At this point, she is constrained ($w = 0$ and $c = y$) in equilibrium, but the Pareto planner is not ($W = w' > 0$). What would the planner do to implement efficiency gains at this point? The planner would make agents share their labor-income risk by providing her with higher consumption in the constrained state but reduce her consumption in future high-income states.[14]

In general, the welfare gains from moving to full commitment are not overwhelming (2.0% if weighted by the ergodic distribution), so it is plausible that families play this equilibrium if commitment is difficult or costly to attain.

4.2.8 *Risk-lovingness* As alluded to before, another form of moral hazard (besides over-consumption) can arise in this model if we allow for a portfolio decision. Value functions can be locally convex for the poor player on the seam between the SS and the OC regions, meaning that the poor player becomes locally risk-loving. The economic intuition for this risk-lovingness is as follows. Suppose the agent could participate in a fair lottery. Then his downside risk would be limited: if he lost the bet and entered the OC region, the rich agent would take on part of the losses by providing transfers eventually. The up-side potential of the bet, however, is enjoyed by the poor agent alone: winning a large amount would bring the economy into the SS region, where the possibility of transfers is remote.

We extend the setting from before by giving players access to two assets: a safe asset with rate of return r and zero variance, and a risky asset with expected rate of return r and standard deviation σ. In addition to consumption and transfers, players now have to choose the portfolio share in the risky asset. The portfolio shares in the risky assets are

[14] For parameter constellations where the transfer region extends all the way to the origin, this type of inefficiency does not occur. One could argue that altruistic agents, especially inside a family, should be able to write contracts that avoid underconsumption of a constrained agent. If we allowed for fully state-contingent contracts between agents, there would be commitment, which we explicitly rule out. Short of fully contingent contracts, one could also imagine restricting altruistic agents to basic lending contracts with a fixed interest rate. This possibility is clearly of interest, but is beyond the scope of the current paper.

functions $z(x) \in [0,1]$ and $z'(x) \in [0,1]$, that is, we rule out short-selling. The resulting law of motion for her wealth is then $\mathrm{d}w_t = (rw_t + y_t - c_t - g_t + g'_t)\,\mathrm{d}t + z_t w_t \sigma\,\mathrm{d}B_t$ and her HJB becomes

$$\begin{aligned}\rho v = {} & \xi\bigl[v(\cdot,\tilde{y}) - v(\cdot,y)\bigr] + \xi'\bigl[v\bigl(\cdot,\tilde{y}'\bigr) - v\bigl(\cdot,y'\bigr)\bigr] \\ & + \alpha u\bigl(c'\bigr) + \bigl(rw' + y' - g' - c'\bigr)v_{w'} + z'^2 w'^2 \frac{\sigma^2}{2} v_{w'w'} \\ & + \max_{c\geq 0}\bigl\{u(c) + \bigl(rw + y + g' - c\bigr)v_w\bigr\} \\ & + \max_{g\geq 0}\bigl\{g[v_{w'} - v_w]\bigr\} + \max_{z\in[0,1]}\Bigl\{\underbrace{z^2 w^2 \frac{\sigma^2}{2} v_{ww}}_{\text{risk-taking}}\Bigr\},\end{aligned}$$

where the portfolio decision shows up in the terms in second derivatives of the value function. The risk-taking decision can be decoupled from the consumption and transfer decision; it is given by the bang–bang-type rule $z^* = I(v_{ww} > 0)$, where $I(\cdot)$ is the indicator function. So she goes fully into the risky asset whenever the value function is convex; otherwise she invests only in the safe asset.

Figure 11 shows the states in which players choose the risky asset (parameters are as in the baseline model). We see that both players invest in the risky asset when caught between the SS and the OC region and "gamble for resurrection." Since the risky asset is assumed to have the same expected return as the safe asset, it is never efficient to invest in it; thus this constitutes a form of moral hazard.

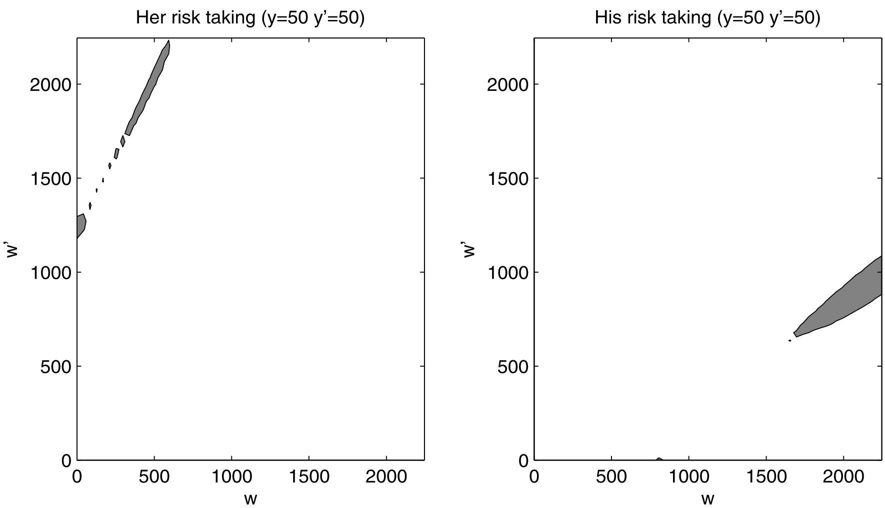

FIGURE 11. Risk-taking regions. For the risk-taking model, the parameters are as in baseline model; see Table 1. Players choose the risky asset inside the gray areas. Not shown are the states where either player has low income.

4.3 *Discussion*

We finally discuss existence and uniqueness of equilibrium and the role that the Brownian wealth shocks play as a randomization device.

We first explain how the transfers-when-constrained equilibrium breaks down when σ is chosen too low. After a certain number of iterations, the transfer motive (μ or μ') of the richer agent turns positive on the seam between the SS and the OC region. The richer player would then prefer to lift the poorer one into the SS region by giving a transfer so as to avoid the OC region and the associated inefficiencies, and the algorithm breaks down because the transfers-when-constrained conjecture is not fulfilled. We experimented with algorithms that allow for transfers within the state space, but were unsuccessful in finding equilibria that converge to stationary policies. Determining transfers within the state space is challenging in our grid-based algorithm because of the bang–bang nature of the transfer decision.[15] We find that problems with the transfer motive occur first on the margins of the grid, where agents' wealth is large and $\{y, y'\}$ are unimportant as a source of income.

Indeed, we have found that the policies in our setting converge to those of an otherwise identical problem without labor income as studied in Barczyk and Kredler (2014), henceforth BK, when players' joint wealth $W \equiv w + w'$ grows large. In their homogeneous setting, the wealth share $P \equiv w/(w + w')$ is the unique state variable and $P \in \{0, 1\}$ are the two steady states in a transfers-when-constrained equilibrium. BK find that a transfers-when-constrained equilibrium does not exist in the absence of wealth shocks. The intuitive reason is that there are tensions between the two players, who disagree on to which of the steady states the economy should converge. These tensions cannot be resolved in a deterministic setting. Mathematically, the ordinary differential equations (ODEs) in P that characterize the equilibrium are overdetermined by the boundary conditions at $P \in \{0, 1\}$. Mixed strategies, the tool of choice to ensure existence of equilibrium in static games, are not tractable in differential games, so BK introduce wealth shocks. They find that a transfer-when-constrained equilibrium then exists, the ODEs being exactly determined.

One may have conjectured that income uncertainty alone would be sufficient for the transfers-when-constrained equilibrium to exist, but we find that this is not the case; it is crucial to have a shock that grows proportionally to joint wealth W.

If shocks to wealth are large enough, then the transfers-when-constrained equilibrium is the unique equilibrium we find computationally by backward iteration. In Table 2, we report the minimal level for σ necessary for this, given different levels of γ and $a \equiv \alpha^{1/\gamma}$. The reparameterization a is a more intuitive measure of altruism, which keeps the generosity of the donor constant when other model parameters change.[16] As mentioned above, we find that the income-process parameters $\{y, y', \xi\}$ have no bearing on

[15] Barczyk and Kredler (2014) discuss one possibility of such a transfer (mass transfers) in more detail in their simplified setting, but can rule out equilibria where such transfers are given to an impoverished agent.

[16] Reparameterization a is the fraction of recipient's consumption to donor's consumption that a donor chooses under power utility. Her first-order condition for transfers is $\alpha c'^{-\gamma} = \alpha(y' + g)^{-\gamma} = c^{-\gamma}$, from which we obtain the optimal consumption ratio $a \equiv (y' + g)/c = \alpha^{1/\gamma}$.

Table 2. Lower bound for σ (in %).

$a = a'$	0.1	0.3	0.5	0.7	0.9
$\gamma = 0.5$	2.5	5.3	3.9*	2.6*	0.9
$\gamma = 1$	1.1	3.2	3.1*	3.1	0.9
$\gamma = 2$	0.8	1.8	2.5	2.4	0.9
$\gamma = 4$	0.4	1.0	1.3	1.6	0.9
$\gamma = 8$	0.1	0.3	0.6	0.7	0.6

Note: Lowest value of σ for which the transfer motive μ stays negative throughout the state space, and thus equilibrium exists for given γ and $a = a'$, where $a = \alpha^{1/\gamma}$ and $a' = \alpha'^{1/\gamma}$. Values are calculated using the homogeneous model that obtains in the limit when wealth becomes large with respect to income; see Barczyk and Kredler (2014). In the cases marked with asterisks, the portfolio-variance effect has to be neutralized for the equilibrium to exist. Values for ρ and r are as in the baseline example. When altruism is asymmetric, we find that the lower bound for σ is governed mainly by the higher altruism parameter.

the issue of existence of the transfers-when-constrained equilibrium. For convenience, altruism is assumed to be symmetric ($a' = a$).[17] We find that for fixed γ, the lower bound for σ is inverse U-shaped in a, just as altruistic-strategic distortions are (not shown here). Tensions disappear at the extremes where the model converges to the selfish or the perfectly altruistic case and are maximal for intermediate values of altruism. When fixing a, the lower bound is (weakly) decreasing in γ. This is due to a portfolio-variance effect that is explained in detail in Appendix A.3.3.

One may choose to think of the noise term as a randomization device such as provided by mixed strategies. Another obvious interpretation is that players save in risky assets whose returns are imperfectly correlated. Examples are shocks to wealth held in housing and other durable goods or entrepreneurial and investment risk. Finally, we may interpret the shocks as disturbances to consumption. Under this interpretation, c is the consumption rate that an agent desires to attain in a given year, but shocks cause realized consumption spending to stray from this desired level. Examples are health expenditures and shocks to prices of commitment goods (i.e., goods the household cannot quickly substitute away from, e.g., renting an apartment or driving a fuel-inefficient car). The levels of σ required for the transfers-when-constrained equilibrium to exist, which are not higher than 3.2% for standard values of γ, seem realistically low in view of these examples. In terms of computation, adding noise smooths value and policy functions and provides additional stability. Mathematically, the noise term is reminiscent of the theory of vanishing viscosity solutions, which is an important tool in the analysis of PDEs, and HJBs in particular.[18]

Finally, we remark upon a potential type of equilibrium that BK find in the absence of shocks. They show that there exists a continuum of *tragedy-of-the commons-type* equilibria, in which players eventually pool their wealth and behave as if they were a perfectly altruistic dynasty (albeit with a higher discount rate than the individual players). In our setting, this type of equilibrium is ruled out by the equilibrium-selection criterion that

[17] When altruism is asymmetric, we find that the lower bound for σ is governed mainly by the higher altruism parameter.

[18] For an excellent introduction to viscosity solutions and the application to control theory, see Bressan's (2010) tutorial. BK also discuss viscosity solutions in differential games in their online appendix.

requires the equilibrium to be the limit of a sequence of finite games. In the final period, imperfectly altruistic players would choose not to pool their wealth, but the richer player would reserve a larger fraction of resources for herself, causing the equilibrium to break down.

5. Testable implications

A novel feature of the model is that the donor conditions transfers on the recipient's wealth and labor income in distinct ways. It imposes fewer restrictions on the *transfer–income derivative* (TID) than previous models do, but adds restrictions in that it distinguishes between transfer–*income* and transfer–*wealth* derivatives (TWD).

The TID is defined as the change in transfers that occurs when the donor's income decreases by \$1 while the recipient's income increases by \$1. In most previously studied altruism models, this derivative can be shown to equal unity, that is, the donor "undoes" the income redistribution by decreasing transfers one-for-one, thereby maintaining the consumption allocation unchanged. Altonji, Hayashi, and Kotlikoff (1997) estimate a TID of 0.13 on cross-sectional transfer data from the Panel Study of Income Dynamics (PSID) and interpret this as evidence against the altruism model. However, as McGarry (2006) argues, the TID restriction does not hold if current income contains new information about future income. Our model can actually generate TDIs both above and below unity, depending on the persistence in agents' income processes.

For wealth changes, the picture in our model is fundamentally different, and we think we are able to point to a new way of testing the restrictions imposed by altruism models in the data. We define the transfer–wealth derivative (TWD) as the change in transfers that occurs when the donor's wealth decreases by \$1 while the recipient's wealth increases by \$1. Due to no commitment, we show that transfer–wealth derivatives (TWDs) are lower than unity if altruism is imperfect. In our numerical example from Section 4.2, we find an average TWD of 0.33 for g and 0.27 for g' (ergodic weights). We refer the reader to Appendix A.4 for calculations and details regarding the TID and TWD.

The fundamental difference between labor income and wealth suggests caution when testing transfer-derivative restrictions. Our model says that it is crucial to differentiate between flow labor income and one-time changes to the stock of wealth, which may not always be easy in real-world data (consider, for example, labor income in the form of a large bonus).

On a different note, the model's implications on consumption behavior are that consumption levels should depend on relatives' assets and labor income in addition to an agent's own assets and income. In families where the wealth distribution is very biased, poor households' consumption should strongly depend on the rich households' resources and only weakly on their own resources. This should be the case even if actual transfers are not observed yet (for families in the OC region). However, this cross-dependence should not be observed for families with a balanced asset distribution (families in the SS region).

There is a large literature that is concerned not with *levels*, but with *changes* in consumption over time, and that tests the implications of the Euler equations on longitudinal data. Our model adds terms to the standard (selfish) Euler equation and predicts that marginal utility should grow fastest when a family is inside the OC region. Empirically, this would become evident in consumption growth being lowest for families with an uneven (but not degenerate) wealth distribution. Drops in consumption should be sharpest when one household begins to receive transfers from another. Consumption growth should be higher in regions where one household receives transfers or where the family's wealth distribution is balanced. Indeed, our model predicts that consumption growth should be the same in these regions as for otherwise similar households that lack family ties to other households.

Finally, the previously mentioned extension of the model that allows for endogenous portfolio choice makes the prediction that households should purchase *lotteries* when their wealth is low relative to that of the rest of the family, especially when they are not yet dependent on transfers from them. This situation corresponds to the seam between the SS and the OC regions in the model; see Figure 11). In reality, we could think of such lotteries as purchasing insufficient insurance for health or longevity, making risky investments in assets or a business, and making risky career choices.

6. Conclusions

This paper provides a building-block model suitable for tackling economic issues in which dynamic behavior of the family is essential. As such, it is an important step in the direction necessary for building more realistic environments for quantitative analysis. A major strength of the framework is that it is consistent with stylized facts on inter vivos transfers and that it delivers precise predictions on the timing of transfers. Thus, an immediate application would be to study the timing and size of inter vivos transfers or remittances.

For future research, the model is suitable as a building block for dynamic models of the (extended) family. Obvious applications are the welfare implications of government policies such as pensions, health insurance, welfare programs, or the estate tax. We argue that a key issue when studying these applications is to acknowledge the interaction of government-provided transfers with transfers already provided by the family.

Some applications will necessitate the introduction of nonmonetary transfers (in-kind, time, co-residence) into our model. In a follow-up project (Barczyk and Kredler (2013)), we extend the current framework using both labor-supply and time-transfer decisions so as to study the macroeconomic consequences of government long-term-care policies. While government-provided care potentially crowds out family-provided care, it may provide households with additional insurance.

Finally, one can view our framework as a theory of partial insurance. The crucial difference between our theory and others in the literature is that impoverished agents receive transfers even if they have never given to the donor in the past and are unable to reciprocate in the future. An example for this is altruistic behavior toward people with terminal diseases or disabilities. From this point of view, it seems promising to further investigate the implications of our theory for risk-sharing. Kaplan and Violante (2010), using a consumption–insurance measure proposed by Blundell, Pistaferri, and Preston (2008), find that the standard Aiyagari–Bewley–Huggett model implies too little consumption insurance against persistent shocks relative to the data, suggesting that households have additional sources of insurance. Family insurance is an obvious candidate to reconcile theory and evidence. Our model provides a framework to test whether family insurance can quantitatively account for the missing sources of insurance.

Appendix

A.1 *Pareto-optimal allocations*

To solve the Pareto planner's problem, individual levels of wealth and labor income can be pooled. Define $W_t \equiv w_t + w_t'$ as the total wealth of the family and, similarly, define $Y_t \equiv y_t + y_t'$ as total labor income.[19]

Given an initial wealth level W_0, the planner chooses consumption policies $\{c_t, c_t'\}$ to maximize

$$\mathbb{E}_0\left[\eta\int_0^\infty e^{-\rho t}\left[u(c_t)+\alpha u\left(c_t'\right)\right]\mathrm{d}t+(1-\eta)\int_0^\infty e^{-\rho t}\left[u\left(c_t'\right)+\alpha' u(c_t)\right]\mathrm{d}t\right]$$

$$\begin{aligned}\text{s.t.}\quad & \mathrm{d}W_t=\left(rW_t+Y_t-c_t-c_t'\right)\mathrm{d}t+W_t\sigma_W\,\mathrm{d}B_{W,t}, \\ & W_t\geq 0\quad \forall t, \text{ all histories,}\end{aligned} \tag{16}$$

where $\sigma_W = \sigma/\sqrt{2}$ and $B_{W,t}$ is a standard Brownian motion. The standard deviation for the planner is smaller than for the individual players, since the planner will allocate one-half of wealth in each risky asset. We will now show that this is the optimal portfolio for the planner.

Since the planner is risk-averse, she should obviously choose the portfolio weights such that the portfolio variance is minimized since both assets have the same expected return. Let P be the fraction of W that goes to her asset and let $(1-P)$ be the one allocated to his asset. Then the law of motion for W is given by

$$\mathrm{d}W_t=\left(rW_t+Y_t-c_t-c_t'\right)\mathrm{d}t+P_tW_t\sigma\,\mathrm{d}B_t+(1-P_t)W_t\sigma\,\mathrm{d}B_t'.$$

The variance is given by

$$\mathbb{E}\left[(\mathrm{d}W_t)^2\right]=P_t^2W_t^2\sigma^2\,\mathrm{d}t+(1-P_t)^2W_t^2\sigma^2\,\mathrm{d}t,$$

which is minimized for $P^*=1/2$. Substituting P^* back into the law of motion yields

$$\mathrm{d}W_t=\left(rW_t+Y_t-c_t-c_t'\right)\mathrm{d}t+W_t\underbrace{\frac{\sigma}{\sqrt{2}}}_{\equiv\sigma_W}\underbrace{\frac{\mathrm{d}B_t+\mathrm{d}B_t'}{\sqrt{2}}}_{\equiv\mathrm{d}B_{W,t}}.$$

[19] Under our assumptions on the endowment processes, Y_t contains enough information to forecast future Y_{t+s}, $s>0$, perfectly. Under more general Poisson processes for y_t and y_t', the planner may need to keep track of y_t and y_t' individually so as to forecast Y_{t+s} correctly. It is straightforward to adapt the arguments here to this case.

$B_{W,t}$ as defined here is standard Brownian motion since it has unit variance, has normal increments, and is serially uncorrelated. This gives us the law of motion stated in the planner's problem (16).

Let $V^{\eta}(W, Y)$ be the value to a planner with wealth W and labor income Y when putting weight η on her lifetime values. The planner's value function satisfies the HJB

$$\begin{aligned}\rho V^{\eta} = \max_{c,c'} \Big\{ & \big[\eta + \alpha'(1-\eta)\big]u(c) + \big[\alpha\eta + (1-\eta)\big]u\big(c'\big) \\ & + \big(rW + Y - c - c'\big)V_W^{\eta} + \xi\big[V^{\eta}\big(\cdot, \tilde{y} + y'\big) - V^{\eta}(\cdot, Y)\big] \\ & + \xi\big[V^{\eta}\big(\cdot, \tilde{y}' + y\big) - V^{\eta}(\cdot, Y)\big] + \frac{\sigma_W^2}{2} W_t^2 V_{WW}^{\eta} \Big\}.\end{aligned}$$

Intratemporal optimality requires that the two margins of allotting consumption to him and to her be equalized:

$$\big[\eta + \alpha'(1-\eta)\big]u_c(c_t) = \big[(1-\eta) + \alpha\eta\big]u_c(c_t) \quad \forall t, \text{all histories.}$$

From this, we obtain the intratemporal optimality condition

$$u_c(c_t) = \frac{(1-\eta) + \alpha\eta}{\eta + \alpha'(1-\eta)} u_c\big(c_t'\big) \quad \forall t, \text{all histories.} \tag{17}$$

As in standard insurance problems, marginal utilities are proportional across states and time. Here, the factor of proportionality is a function of the planner's weight η on her and the altruism parameters α and α'. It is instructive to consider the extreme cases where $\eta = 0$ or $\eta = 1$. Placing all weight on her yields $u_c(c_t) = \alpha u_c(c_t')$, whereas placing all weight on him yields $u_c(c_t) = \frac{1}{\alpha'} u_c(c_t')$. Thus, just as in the static altruism model, the ratio of marginal utilities is restricted to the interval $[\alpha, \frac{1}{\alpha'}]$. The more altruistic are both agents, the smaller is the consumption inequality a Pareto planner may tolerate. For perfect altruism ($\alpha = \alpha' = 1$), there is a unique Pareto-optimal allocation and both agents consume the same amount always. As altruism goes to zero, the bounds $[\alpha, \frac{1}{\alpha'}]$ approach zero and infinity, until reaching the standard case with selfish agents.

When using the functional form of power utility in Equation (17), we see that the planner will choose the consumption rate of him as a fixed proportion of her consumption rate. So given that the planner wants to devote expenditures $C_t \equiv c_t + c_t'$ on both players' combined consumption in a given state, it is now easy to determine how consumption should be split between the two agents. From this rule, we can then write an indirect utility function of the form $U_{\eta}(C_t) \equiv \frac{H_{\eta}}{1-\gamma} C_t^{1-\gamma}$ for the planner, where H_{η} is a constant that depends on η. Since U_{η} represents the same preferences for all η, this implies that the planner will choose the same aggregate consumption plan $\{C_t\}$ regardless of η; only the division of C_t between the agents will depend on η. Furthermore, it implies that the planner always runs down aggregate wealth K_t at the same rate in any efficient allocation. The reasoning just laid out leads us to the following solution strategy for the planner's problem(s). First, solve a standard Bewley problem in $\{C_t\}$ for a planner who faces constraints of the form in (16); then find the two agent's consumption plans $\{c_t, c_t'\}$ given $\{C_t\}$ according to the sharing rule implicit in (17).

We now turn to the intertemporal optimality conditions. Note that Equation (17) gives us c_t' as a function of c_t,[20] so that the planner's problem collapses to a conventional consumption–savings problem with a modified objective function (to see this, substitute out c_t' in the objective (16) using (17)). This yields the intertemporal condition (10), which we reproduce as

$$\underbrace{\mathcal{A}u_c(c)}_{\text{expected growth of } u_c} \leq (\rho - r)u_c(c).$$

This is the same Euler equation as in the standard one-person consumption–savings problem. If this Euler equation did not hold, the planner should reallocate resources intertemporally for one agent maintaining the present value of resources allocated to her/him constant. This equation says that marginal utility has to grow at the expected rate $(\rho - r)$ whenever the planner is unconstrained (i.e., $W_t > 0$), but may grow at a lower rate when the planner is constrained.

As was pointed out in the Introduction, all commitment models of the family (such as the unitary and collective model) build on the above Pareto problem.

Even though equilibrium allocations are usually not efficient in our framework, it is still instructive to think about arrangements that would implement efficient allocations. In a deterministic setting without labor income, Barczyk and Kredler (2014) show that any efficient allocation can be implemented by assigning appropriate shares of initial assets to the agents and then shutting down transfers forever. In the setting of the current paper, it turns out that an analogous arrangement can support any efficient allocation in the case that $\sigma = 0$ and $u(\cdot)$ is homothetic.

This analogous arrangement is as follows. Players commit to share their joint flow labor income $Y_t = y_t + y_t'$ for all t and initial assets $W_0 = w_0 + w_0'$ according to a fixed sharing rule, that is, she receives a fixed fraction $\tilde{\eta}$ of Y_t and he receives the rest (i.e., $(1 - \tilde{\eta})Y_t$). Furthermore, transfers are ruled out for all t. If utility $u(\cdot)$ is homothetic, then agents' consumption rules will equal the planner's rule since the agents' problems are just a scaled version of the planner's problem. If $\sigma = 0$, indeed, this mechanism yields exactly the same consumption allocation as the planner's problem since there are no gains from insuring players against investment risk. If $\sigma > 0$, however, there are also gains from portfolio diversification. The planner will then hold exactly half of total assets in each account and has to rely on transfers to implement the efficient allocation, causing the equivalence to the resource-sharing mechanism to break down.

A.2 *Transfers decision when both players are broke*

The case where both players are broke introduces the additional complication that also the donor faces a constraint. Specifically, the donor's consumption plus transfers cannot exceed labor income. However, the different cases and the intuition are very similar to the case where only the recipient is constrained.

[20]When $\eta = 1$ and $\alpha = 0$, Equation (17) does not give us c_t' as a function of c_t any more; in this case, however, it is obviously optimal for the planner to set $c_t' = 0$ for all t. Analogously, $\eta = 0$ and $\alpha' = 0$ imply that $c_t = 0$.

Consider the case $w = w' = 0$. We will consider the situation where her flow labor income is higher than his: $y > y'$. We will first study her transfer decision given that $g' = 0$ and later verify that indeed $g' = 0$. We assume that $\mu < 0$.

Her problem is to maximize $H(c, g)$ in (11) subject to the additional constraint

$$c + g \leq y, \tag{18}$$

which says that she cannot spend more than her current labor income.

Obviously, if the unconstrained maximizers (c_0, g_{unc}) from Section 3.3 fulfill this constraint, then they are also the solution to the constrained problem. Also, the intuition from the unconstrained case applies.

From now on we will be concerned with the case where the constraint binds, which means that the unconstrained maximizers (c_0, g_{unc}) fall *outside* the feasible set defined by (18) and the nonnegativity constraints $c \geq 0$ and $g \geq 0$. We will now show that the optimal choice must fulfill her budget constraint with equality and that transfers do not go into savings. Formally, we want to show that the constrained maximizers $(c_{\text{constr}}, g_{\text{constr}})$ lie on the line segment D defined by

$$D \equiv \{(c, g) : c + g = y, g \leq c'_0 - y'\}.$$

We will first show why $g \leq c'_0 - y'$ must hold. If we consider a pair (c, g) such that $g > c'_0 - y'$, a (feasible) decrease in g always increases H since $\partial H/\partial g < 0$ in this region (recall that he is saving the marginal transfer unit, which she dislikes). Thus we must have $g_{\text{constr}} \leq c'_0 - y'$.

Second, to see why $c + g = y$ must hold with equality, note that the objective function H is strictly concave in (c, g) in the region where $g \leq c'_0 - y'$ by strict concavity of u. So any interior pair $\{(\tilde{c}, \tilde{g}) : \tilde{c} + \tilde{g} < 0, \tilde{g} \leq c'_0 - y'\}$ must be dominated by any convex combination of $(\tilde{c}, \tilde{g})$ and the unconstrained maximum (c_0, g_{unc}) since $H(\tilde{c}, \tilde{g}) \leq H(c_0, g_{\text{unc}})$ and H is strictly concave (note that $c_0 \geq 0$, $g_{\text{unc}} \geq 0$ and $g_{\text{unc}} \leq c'_0 - y'$ by definition of g_{unc}). Since $(\tilde{c}, \tilde{g})$ was interior, there thus must always be possible improvements and $(\tilde{c}, \tilde{g})$ cannot be optimal.

We have now established that the constrained maximizers must lie on the line segment D. Since this is the case, we can reduce the problem of maximizing (11) subject to (18) to a simpler auxiliary problem in one choice variable by writing

$$\max_{g \in [0, c'_0 - y']} \{u(y - g) + \alpha u(y' + g)\}. \tag{19}$$

Here, we recognize a static altruism problem subject to the additional constraint $g \leq c'_0 - y'$ (which comes from the fact that she never wants to give transfers that flow into savings); we see that apart from this upper bound on transfers, the problem is now independent of the dynamic aspects of the game, that is, the derivatives of the value functions $v_w, \ldots$.

Note that the function to be maximized, $\tilde{u}(g) \equiv u(y - g) + \alpha u(y' + g)$, is strictly concave (again by strict concavity of u) and the maximand is chosen from a closed interval.

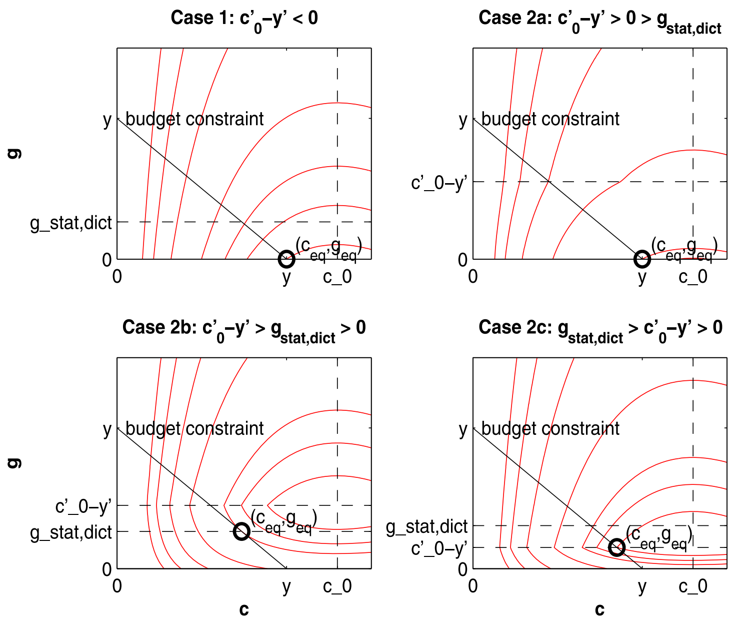

FIGURE 12. Her transfer decision when $w = w' = 0$.

It will thus be possible to characterize the solution by just checking whether the unconstrained maximizer of $\tilde{u}(g)$ falls into the constrained set. We define

$$g_{\text{stat,dict}} = \arg \max_{-\infty<\tilde{g}<\infty} \{u(y-\tilde{g}) + \alpha u(y' + \tilde{g})\},$$

where the subscript stat, dict suggests that this is the transfer that she would choose in a static problem if she could also force negative transfer upon him. For CES utility, we obtain $g_{\text{stat,dict}} = (\alpha^{1/\gamma} y - y')/(1+\alpha^{1/\gamma})$.

The following cases can arise in the situation where her constraint binds. Figure 12 depicts the level lines of her Hamiltonian as curves, the constraint as a line, and the optimal choice as a circle.

1. $c'_0 < y'$: He saves even when transfers are zero, so she should set the transfer to zero since $v_{w'} < v_w$. The transfer margin is dominated by the consumption margin.

2. $c'_0 \geq y'$: The following subcases can arise:

(a) $g_{\text{stat,dict}} \leq 0$: She sets $g_{\text{constr}} = 0$ since the criterion in (19) is decreasing in g for all $g \in D$. The transfer margin is dominated by the consumption margin.

(b) $0 < g_{\text{stat,dict}} < c'_0 - y'$: There is an interior solution and she sets $g_{\text{constr}} = g_{\text{stat,dict}}$; the unconstrained maximizer for the problem (19) falls into the feasible set. Here, she sets the transfer margin equal to the consumption margin and the allocation is the one predicted by the static altruism model.

(c) $g_{\text{stat,dict}} > c'_0 - y'$: A corner solution occurs in the auxiliary problem (19). She increases transfers until reaching his satiation point where he would start to save the marginal transfer. She stops the transfer at this point and sets $g_{\text{constr}} = c_0' - y'$. Her transfer margin is *lower* than the consumption margin when considering a marginal *increase* in transfers at the optimum, but the transfer margin is *larger* than the consumption margin when considering a marginal *decrease* in the transfer at the optimum. This is what the kinks in the level lines of the Hamiltonian indicate.

It is important to recall that we were always operating under the assumption that the unconstrained maximizer (c_0, g_{unc}) does not fall into feasible set, so she sets savings to zero. This means that her consumption margin is larger than her savings margin in all cases.

We can summarize the solution for all (constrained) cases by

$$g_{\text{constr}} = \max\{0, \min\{g_{\text{stat,dict}}, c_0' - y'\}\}.$$

Since the budget constraint always holds with equality when she is constrained, her consumption is always given by $c_{\text{constr}} = y - g_{\text{constr}}$. His consumption is $c'_{\text{constr}} = y' + g_{\text{constr}}$ (recall that she never gives transfers that would go into his savings, so we need not enforce the upper bound c_0').

Finally, we can also conclude that when both players are bankrupt, then only the player with the higher flow labor income (the "labor-income-rich" player) can possibly give transfers. Note that $g_{\text{stat,dict}}$ is negative for the labor-income-poor player, so it is a dominant strategy for him to set transfers to zero. For this to hold, we technically need the altruism parameters to be lower than 1: if α was larger than 1, then she might want to transfer to him although she has lower flow labor income than he does.

A.3 *Comparative statics and robustness*

In general, our algorithm performs well and is stable across the parameter space. The qualitative features of equilibrium discussed in Section 4.2 are always preserved under imperfect altruism.

A.3.1 *Special cases* For both selfishness ($\alpha = \alpha' = 0$) and perfect altruism ($\alpha = \alpha' = 1$), the equilibrium is described by a Bewley-type economy. In the selfish case, both agents solve their separate consumption–savings problem and transfers are zero. The agents' consumption depends solely on their own state (w, y) and is invariant in the other agent's state (w', y'). When altruism is perfect, a dynastic household solves a Bewley problem given the agents' joint labor income resources (as in the planner's problem in Appendix A.1). Agents' consumption is equalized in all states and depends solely on the joint state (W, Y), but depends neither on the distribution of assets w/W between agents nor agents' contribution to total labor income y/Y.

We find that the equilibrium consumption functions converge to these special cases when (α, α') approach their bounds. Consumption depends more on joint wealth when altruism is high and more on own wealth when altruism is low. The transfer regions and OC regions become large when $(\alpha, \alpha') \to (1, 1)$; they become small and start at higher levels of (w, w') when $(\alpha, \alpha') \to (0, 0)$. In both cases, distortions to consumption–savings decisions vanish as the limit is approached; distortions and overconsumption are most pronounced for intermediate values of (α, α'). Immiseration occurs ever faster in the OC region when $(\alpha, \alpha') \to (1, 1)$ as consumption of the poor agent gets close to the rich agent's consumption level.

Further, one-sided and symmetric altruism warrant discussion. Importantly, strategic interactions between players are still present for these cases.

Even when altruism is one-sided, the other player's state (w', y') is of interest for both players. A selfish agent will contemplate her counterpart's situation so as to gauge the likelihood of transfers, and an altruistic agent has to keep track of the selfish counterpart's state so as to see if transfers are needed. As is to be expected, in equilibrium, the selfish player never gives transfers. There is neither a transfer nor an OC region on the side where the selfish player has high wealth. On the side where the altruistic agent has high wealth, there is a transfer region and an OC region with the characteristics described before.

In the case of symmetric imperfect altruism ($0 < \alpha = \alpha' < 1$), the only simplification that occurs is that the equilibrium is now symmetric. Its features are as in the baseline example. Apart from this, symmetric imperfect altruism is not fundamentally different from asymmetric altruism. Note that agents do not have the same preferences: they still disagree on the desirability of allocations—preferences are only mirror-symmetric. The only case in which agents agree is perfect altruism.

A.3.2 *Changes in* γ Due to our parsimonious specification of preferences, γ plays various roles. In addition to governing risk aversion and the elasticity of intertemporal substitution, it also impinges on transfer behavior.[21] In our setting, however, as is apparent from the definition of a, γ only affects transfer behavior in conjunction with α. It turns out that when keeping (a, a') constant, the equilibrium transfer functions are essentially invariant in γ. Distortions to consumption–savings decisions are not visibly affected either. Qualitatively, the consumption functions also preserve their characteristic. The only way we find γ to affect the equilibrium is through the well known precautionary-savings mechanism. The higher is γ, the stronger is the precautionary-savings motive and the larger is agents' wealth under the ergodic distribution. Since more time is spent in regions with high wealth, this also means that transfers flow more often.

A.3.3 *Changes in* σ It turns out that there is also an upper bound on σ above which equilibrium ceases to exist for given (γ, a), which we report in Table 3. The intuition for this upper bound is the following. Recall from the planner's problem (Appendix A.1) that the optimal portfolio choice of the planner is to keep half of total wealth W in each agent's account—this minimizes portfolio risk. So when agents are perfectly altruistic,

[21]If transfers had a price, then γ would also determine the elasticity of transfers with respect to this price. In our setting, both players face the same prices, so this is irrelevant. An example where the transfer elasticity is potentially important is the case of remittances. Consider an emigrant who sends funds back home. Then the transfer elasticity governs the response of remittances to changes in the real exchange rate between the host and the home country. A specification of preferences that decouples the transfer elasticity from risk aversion and intertemporal substitution is $\mathbb{E}_0 \int_0^\infty e^{-\rho t}[c_t^\beta + \alpha c_t'^\beta]^{(1-\gamma)/\beta}\,dt$, where β governs the transfer elasticity.

Table 3. Upper bound for σ.

$a = a'$	0.1	0.3	0.5	0.7	0.9
$\gamma = 0.5$	10.6	7.6	–	–	1.2
$\gamma = 1$	25.0	8.8	–	3.5	1.3
$\gamma = 2$	5.8	4.9	4.3	3.1	1.2
$\gamma = 4$	3.4	3.2	3.0	2.6	1.3
$\gamma = 8$	2.0	2.1	1.9	1.7	1.1

Note: Highest value of σ for which the transfer motive μ stays negative throughout the state space and thus equilibrium exists (given γ and $a = a'$). Values are calculated using the homogeneous model that obtains in the limit when wealth becomes large with respect to income. ρ and r are as in the baseline example. In the cells with dashes (–), the equilibrium does not exist for any value of σ unless the portfolio-variance effect is neutralized.

they will distribute their assets in this way. However, for imperfect altruism, the equilibrium dynamics are such that the poorer agent heads toward being broke. At that point, however, portfolio risk is maximal from the family's perspective, which creates an incentive to head back to a more balanced asset distribution. When altruism gets close to perfect, value functions flatten out because agents become indifferent with respect to the distribution of wealth. The portfolio effect then at some point overrides the richer agent's preference for own wealth, making the transfer motive positive and thus precluding the transfers-when-constrained equilibrium. Since the portfolio effect is increasing in σ, this occurs more often for high values of σ, thus inducing an upper bound for σ for a given (γ, a) combination. For some parameter configurations, the upper bound for σ cuts below the lower bound, so that there is no level of σ under which equilibrium exists for given (γ, a). The dashes in the table indicate when this occurs.

However, we note that the portfolio effect is not in the spirit of why we introduced shocks to assets in the first place—to overcome the underlying tensions in the setting through an element of uncertainty. When neutralizing the portfolio effect, we can always find σ large enough to ensure of equilibrium for any (γ, a). The lower bound on σ persists, but the upper bound disappears. We thus argue that the upper bound on σ is a less serious limitation for the usefulness of our model than the lower bound.

From Table 3, we see that the upper bound on σ is decreasing in a. Value functions become flatter as altruism increases, and it becomes easier for the portfolio effect to override agents' preference for own assets, causing transfer motives to turn positive. The bound is for the most part decreasing in γ: the more risk-averse agents become, the stronger is the portfolio effect.

We can think of two ways to neutralize the portfolio effect while maintaining the smoothing force of shocks, the first of which we have implemented.

The first is to assume that only the asset distribution $P = w/(w + w')$ between agents is subject to disturbances, but total family assets $W = w + w'$ are unaffected by shocks. Technically, this approach is paramount to finding a viscosity (i.e., smooth) solution to the HJBs in the cake-eating setting studied by Barczyk and Kredler (2014). Indeed, we find that there is no upper bound on σ when we follow this approach. The reason we did not choose this modeling approach in the first place is that it lacks a microfoundation: there are very few real-world examples of such shocks (an example is a family receiving a court verdict that one family member has property rights on an asset previously thought to be the property of another).

A second, more microfounded, way of shutting down the portfolio effect would be to make shocks proportional to consumption c and not assets w. This would also be in line with some of the examples we gave to motivate our shocks to assets (e.g., expenditure shocks, such as repair of consumer durables, or health shocks). To see why this approach is likely to work, observe that for perfectly altruistic agents, consumption functions are invariant in P. Thus expenditure risk would be invariant in P, canceling the portfolio effect. The problem with this approach is that it is unclear how such consumption shocks should be handled technically when an agent is constrained: Suppose both agents are broke and are hit by a negative expenditure shock they cannot pay for (this may occur since Brownian-motion shocks are unbounded). It is not straightforward to handle this situation, so we chose not to take this avenue.

A.3.4 *Other parameters* Changes to the remaining parameters in the model have the expected effects. An increase in ρ makes agents more impatient and brings consumption functions to a higher level, but leaves the qualitative features of the equilibrium unchanged. The opposite is true for an increase in r. As mentioned before, widening the gap between agents' flow labor income y and y' results in transfer regions drawing closer to the origin and eventually reaching it.

We have also computed equilibria for the case where only she faces income risk. In this case, only the player with income risk engages in precautionary-savings behavior. He, however, ends up broke eventually almost surely (we set $\rho < r$). The ergodic distribution collapses to a "mass strip" on the line where he is broke, with a mass point at the origin.

A.4 *Transfer derivatives*

A.4.1 *Transfer-income derivatives* To compute transfer-income derivatives (TIDs), we compute the change in transfers Δg following a downward jump in the donor's labor income paired with an upward jump in the recipient's labor income of the same magnitude (conditional on transfers still being positive after the change to labor incomes). Dividing the ensuing change in transfers Δg by the change in income $\Delta y = y_2 - y_1$ gives us the analog of the TID to a model where income can be varied in a continuous fashion. Our model generates TIDs below and above unity, depending on the hazard rates of labor-income changes for agents. This softens the restrictions from the one- and two-period models considered so far in the literature, which impose that the TID equal unity.

The intuition is that in situations where a donor gives transfers, he essentially dictates the allocation, taking into account total dynasty wealth. If the hazard rates for both labor income processes are the same, then an increase in his labor income paired with a decrease of her labor income of the same size leaves the dynasty's lifetime wealth unchanged, and the donor implements the same consumption policies as before. For this special case, we find a TID very close to unity for states where transfers flow (e.g., in our baseline example). When considering asymmetric hazard rates, we are able to generate TIDs that differ from unity in both directions. This is because changing both labor incomes by the same amount has different implications on the dynasty's permanent labor income. To see this, consider the situation where the agent with the more persistent labor-income process draws a bad shock while the agent with a less persistent process draws a good shock. This is definitely bad news for expected total dynasty income and the donor will decrease transfers. This is similar to the mechanism in McGarry (2006), except that her model yields TIDs strictly below unity.

A.4.2 *Transfer-wealth derivatives* Suppose that an econometrician measures the cumulative transfers over a time interval Δt (say a month or a year). At a given point $x_0 = (0, w_0'; y_0, y_0')$ in the state space, he gives transfers $g'(x_0)\Delta t > 0$ to her over this time interval (to a first order). We will now study how the observed transfers change when the donor's initial wealth is decreased by Δw, while the recipient's initial wealth is increased by Δw. We assume that Δw is small enough so that some transfers still flow over Δt when starting the economy from the new endowment point $(\Delta w, w_0' - \Delta w; y_0, y_0')$ (assuming that there are no changes to labor income).

Denote by $c_{\lim}(x_0) = \lim_{w \searrow 0} c'(w, w_0'; y_0, y_0')$ her consumption when given a small amount of assets. The law of motion for her wealth at the new endowment point $(\Delta w, w_0' - \Delta w; y_0, y_0')$ is given by $\dot{w}' = y_0' - c_{\lim}(x_0)$, to a first order. This means that it will take a time interval $\tilde{\Delta t} \equiv \Delta w / [c_{\lim}(x_0) - y_0']$ until she runs out of wealth. She then starts to receive a transfer $g'(x_0)$, so that the total transfer received over Δt is given by $g'(x_0)(\Delta t - \tilde{\Delta t})$. Note that the probability of labor-income changes is of lower order and may be dropped to a first approximation.

The transfer-wealth derivative (TWD) at x_0 is then defined as the change in transfers (observed over Δt) divided by the change in wealth Δw,

$$\begin{aligned}\mathrm{TWD}(x_0) &\equiv \frac{(\Delta t - \tilde{\Delta t})g'(x_0) - g'(x_0)\Delta t}{\Delta w} = \frac{-g'(x_0)}{c_{\lim}(x_0) - y_0} \\ &= \frac{-1}{1 + \dfrac{c_{\lim}(x_0) - c(x_0)}{g'(x_0)}} > -1,\end{aligned}$$

where the last step uses the fact that $c(x_0) = y_0 + g'(x_0)$.

We see that the larger is the recipient's drop in consumption upon going broke (i.e., the drop in consumption upon going broke), the lower is the TWD. We also see that the TWD is less than unity in absolute value, unlike in static models. The intuition for this may be gleaned from the following example: say he is rich and provides transfers of \$10,000 per year to her. She has labor income of \$10,000 per year, so her consumption is \$20,000. The standard altruism model would then predict that if we transfer \$10,000 from him to her in the beginning of the year, he would choose transfers equal to zero and both would consume the same as before. We see that transfers react one-to-one to a redistribution of wealth. In our model, however, she will overconsume once she has received \$10,000 since she knows that he will provide transfers later in the year once she becomes broke. If the consumption discontinuity is such that she spends at a rate of \$30,000 per year, her wealth will decline at a rate of \$20,000 per year so that she becomes broke after 6 months. He will then provide another \$5,000 of transfers over the second half of the year. As can be seen in this example, the TWD is one-half: transfers decrease by $5,000 following a wealth redistribution of $10,000. The prediction is different because our model is dynamic; in a one-period model, the donor can essentially commit to a transfer over the duration of the model period.

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Submitted March, 2013. Final version accepted November, 2013.