Blast from the past: The altruism model is richer than you think

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October 15, 2021

We provide the first full theoretical characterization of the standard two-period altruism model in a deterministic setting, thereby providing a coherent methodology for studying equilibria in this class of models. Under general conditions, policy and value functions are discontinuous; non-smoothness escalates one order higher than in one-player settings due to differing interests among players. We show that there exists a novel type of front-loaded transfer that enables the recipient to stay in autarky. The logic behind it challenges standard theory as it severs the link between transfer motives and first-order conditions. Our results revise qualitative predictions of dynamic altruism models and highlight that their computation demands global, not local, methods. Our results are robust to introducing income shocks and to varying assumptions on parent savings. Numerical experiments indicate that the correctly solved model features more front-loading of transfers than previously thought, accompanied by higher savings of transfer recipients.

JEL codes: C73, D15, D64, E21
Keywords: consumption-saving decisions; inter-vivos transfers; altruism; savings games; non-smooth equilibria

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1 Introduction

Altruistic preferences à la Becker (1974) are a popular tool in modeling interaction among family members, but also among other economic agents. However, as we will show in this paper, the current state of knowledge is incomplete in ways that matter for predictions and computation.

The interaction of altruistic agents in *static* settings is very well-understood and delivers a set of intuitive predictions. All of them are subsumed in the donor's first-order condition (FOC)

$$u'(\underbrace{y_p - g}_{=c_p}) \geq \alpha u'(\underbrace{y_k + g}_{=c_k}) \qquad \begin{cases} \text{with} = & \text{if } g^* > 0, \\ \text{with} \geq & \text{if } g^* = 0, \end{cases} \tag{1}$$

where $u(\cdot)$ is a standard increasing and concave felicity functional, y_i is agent i's wealth, p being the parent (donor) and k being the child ("kid", recipient), $g \geq 0$ is the parent's transfer and $\alpha > 0$ is a parameter measuring the parent's altruism. Directly from this FOC, the following testable predictions follow: (P1) transfers, g, are increasing in the donor's wealth, y^p; (P2) transfers are decreasing in the recipient's wealth, y^k; (P3) redistributing one unit of wealth from a donor (i.e. a parent choosing $g^* > 0$) to the recipient leads to a one-unit decrease in transfers, leaving the consumption allocation $\{c_p^*, c_k^*\}$ unaffected.

When moving to more realistic life-cycle settings, the literature aims to generalize the FOC (1), and with it predictions P1-P3, to dynamic settings. When assuming that family members can commit to future actions,[1] the FOC (1) indeed continues to characterize the consumption allocation, both across time and states of the world. Also predictions P1-P3 carry over, where "transfers" and "wealth" have to be replaced by "present value of lifetime transfers" and "present value of lifetime income plus initial wealth". However, the commitment assumption has the important shortcoming that it leaves the timing of transfers indeterminate. Intuitively, only the present value of transfers matters; it is irrelevant who carries the family's wealth since contracts – and thus trust among family members – are perfect. A second downside of the commitment assumption is that much of the literature deems it unrealistic.

When removing the commitment assumption, an important and pervasive result surfaces – the *Samaritan's Dilemma* –, but a new obstacle arises – multiple equilibria. The Samaritan's Dilemma (e.g. Buchanan, 1975; Bruce & Waldman, 1990) is the feature that a future transfer disincentivizes child savings, leading to inefficiency. The altruistic parent is trapped. Multiple equilibria occur when parent and child make their savings decisions simultaneously, as shown by Lindbeck & Weibull, 1988 in a two-period setting. Intuitively, at intermediate levels of child wealth, agents may either coordinate on a "virtuous" equilibrium in which both agents correctly expect the child

[1] This is a typical assumption in collective models, see Mazzocco (2007).

to live in autarky in the final period and the child chooses high savings, or a "rotten" equilibrium in which the child saves little and expectations are coordinated on the child receiving help in the final period.

To operationalize the altruism model in quantitative work, a consensus has thus formed to solve discrete-time altruism models as sequential-move (Stackelberg) games. The seminal paper is Altonji et al. (1997) (AHK), who assume that the parent is the Stackelberg leader in a two-period game.[2] They assume that the parent first chooses consumption, savings and a transfer to the child, after which the child makes its consumption-savings decision.

There are three main reasons for the popularity of the AHK framework in applied work. First, the sequential timing of decisions yields a unique outcome. Second, AHK provide a proof (based on first-order conditions) that the model with uncertainty over the child's income and liquidity constraints pins down the timing of transfers, arguing that transfers in the first period can only flow if the child is liquidity-constrained (we term this prediction *P4*). The intuition is the Samaritan's Dilemma: In the parent's view, the child over-consumes (and under-saves) if given the choice to do so, thus "spoon-feeding" the child an appropriate amount in the initial period is optimal. Third, AHK's results seem to justify the use of FOC (1) in dynamic settings, thereby generalizing predictions P1-P3.

Here is where our results come in. AHK's proof implicitly assumes continuity and differentiability of value functions. However, we show that value and policy functions have multiple kinks (i.e. points of non-differentiability) and even discontinuities, which invalidates the use of first-order conditions. Our baseline model is a simplified version of AHK in which we strip out the parent's savings decision and the uncertainty over the child's income. This simplifies the analysis substantially but leaves intact all qualitative model features, as we show by extending the baseline model. Interestingly, eliminating the savings choice of the parent *does not* remove strategic considerations for the parent, which is what some of the literature had conjectured.

In line with what is commonly believed, we find that there is a (essentially) unique equilibrium that obtains by backward induction and that pins down the timing of transfers. Discontinuities in value and policy functions arise in the initial period. At intermediate levels of child wealth, the child's savings correspondence jumps from a "rotten" local optimum (with future transfers) to the "virtuous" local optimum (featuring autarky). While the child is indifferent at the switching point, the parent strictly prefers the virtuous outcome, which manifests itself as a discontinuity in the parent's value function. Stepping backwards one stage, the parent wants to capture this discrete increase in value by nudging the child into autarkic savings with a transfer. We call this novel type of transfer a *shot to autarky*, which we show must occur under weak conditions.[3] Moreover,

[2]Recent papers following AHK's timing protocol are Kaplan (2012), Boar (2020) and Chu (2020), more on these below.

[3]Interestingly, shots to autarky occur in the same region of the state space for which Lindbeck & Weibull (1988)

we find that another novel type of transfer occurs under some parameterizations, at somewhat lower levels of child wealth: The parent gives an initial-period transfer to a child that engages in Samaritans-Dilemma-type savings. Finally, when the child starts the game with low wealth, we find the type of transfer that is characterized by the FOC (1) and identified by previous literature: *spoon-feeding* transfers to constrained children.

What is absolutely crucial is that in the correctly-solved model, only *one* type of transfer (spoon-feeding) is characterized by the FOC (1). Shots to autarky follow a profoundly different logic: When lifting the child to autarky, the parent actually induces the child to consume *less* —and save more— than it would have in the absence of the transfer. Thus, the *operativeness* of transfer motives (i.e. transfers being positive, which is a fundamental notion in the literature since Barro's, 1974, influential work), is not equivalent to the parent's spoon-feeding FOC (1) holding.

The richness of the correctly-solved altruism model has the downside that it is more challenging to formulate straightforward testable predictions. Since transfer and consumption functions are discontinuous and non-monotone in the first period, predictions P1-P3 break down. Similarly, shots to autarky are incompatible with P4. Closer inspection of the equilibrium reveals the following testable, yet less straightforward, predictions: (P5) The child's consumption growth and the child's savings correlate positively with the front-loading of transfers. (P6) Initial-period transfers are U-shaped in children's wealth relative to parent wealth (after transfers), conditioning on positive transfer realizations. In our opinion, however, the ultimate test for altruism models – when properly solved and calibrated – will be if they can help us make sense of the data on consumption, savings, inter-vivos transfers and bequests.

We show that our main results are robust to (1) also allowing the parent to save and (2) to introducing uncertainty by assuming that the child is subject to income shocks. When shocks have continuous support, some smoothing occurs but discontinuities remain present, even for unrealistically large income risk. When shocks have discrete support, as under a discrete-state Markov process, no smoothing occurs at all; in fact, additional kinks and discontinuities are introduced.

Our results imply the following recommendations for the numerical solution of discrete-time altruism problems: i) Optimization algorithms should allow for multi-peaked and discontinuous criteria, where global optima may fall on points of discontinuity.[4] ii) To compute expectations, integration methods should be used that are able to deal with discontinuities.[5] iii) Value func-

find multiple equilibria. While the simultaneous-move setting delivers no clear prediction which regime is played in this region, in the sequential-move setup the parent uses her first-mover advantage to nudge the economy into her preferred regime: autarky.

[4]Brute-force grid search always works, but is of course computationally demanding. Shocks with continuous support that occur right after a decision render the criterion smooth and hence allow for optimization algorithms that rely on FOCs (recalling that FOCs are only necessary, but not sufficient, since the criterion can be multi-peaked).

[5]Brute-force summation over a fine grid and Monte-Carlo integration are appropriate, but quadrature methods are not since they rely on value functions being well-represented by (smooth) polynomials.

tions should be interpolated by the nearest-neighbor method to allow for discontinuities; linear, polynomial and spline interpolation should only be used if a continuous-support shock smooths value functions in the stage right after the approximation is taken. Our solutions for the case of power utility (see Appendix B) provide a benchmark for evaluating the precision of computational algorithms.

Finally, one may now ask: How far off would we be if we did not solve the altruism model correctly (*global method*) but relied on first-order conditions (*local method*), as previous literature did? Our tentative answer is "Quite a bit", which we obtain by solving our model using a coarse calibration of an income process across two generations (see Section 5). Qualitatively, we find that the local method goes wrong for children who are similar to their parents in current income but have low future income. Quantitatively, the correctly solved model generates fewer families in which the Samaritan's Dilemma plays out, replacing these by autarkic outcomes such as shots to autarky. This leads to significantly higher savings and more front-loaded transfer patterns under the correct solution.

Our paper contributes to theoretical, empirical, and applied literatures.

In terms of theory, our paper is most closely related to papers that study dynamic altruism settings without commitment, such as Lindbeck & Weibull (1988), discussed above, and Bruce & Waldman (1990).[6] In our own work, we have studied altruism models in continuous time. In most of this work (Barczyk & Kredler, 2014*a*, Barczyk, 2016, Barczyk & Kredler, 2018), we use Brownian shocks that together with the continuous-time assumption are sufficient to smooth value functions such that all transfers are of the spoon-feeding type. In Barczyk & Kredler (2014*b*) we study a deterministic infinite-horizon game in continuous time and find that when restricting attention to Markovian strategies, no shots to autarky can occur. This result is compatible with our results here since shots to autarky are non-stationary in nature: the parent provides such a transfer only in the initial but not the final period. This strategy, however, is not Markovian in an infinite-horizon setting with wealth as the unique state variable.[7]

A sizeable literature attempts to empirically disentangle motives for financial transfers among family members. Cox (1987) and Cox & Rank (1992), for example, argue their data is more consistent with exchange than altruism. However, they rely on predictions P1-P4, which we show

[6]Bruce and Waldman study sequential decisions in a deterministic two-period model, but they add a child action that can increase the child's income in detriment of the parent's income (as in Becker's Rotten-Kid Theorem). As part of the discussion they conjecture that there is an equilibrium where first-period transfers lead to efficient savings by the recipient. Their arguments rely on continuity of policies and first-order conditions, hence they may only characterize a subset of equilibria.

[7]The no-commitment assumption is crucial here. In a dynamic altruism model with full commitment to future transfers, shots to autarky (with no subsequent transfers) are typically an equilibrium. But shots to autarky are difficult to sustain as equilibria if commitment is absent. Intuitively, the recipient may consume up her wealth and ask for transfers again, which then the altruistic donor cannot deny. The donor therefore refrains from the shot to autarky in the first place, a phenomenon that Barczyk & Kredler (2014*b*) call the Prodigal-Son Dilemma.

to be violated. Similarly related is the empirical test of the transfer-income derivative (P3) by AHK (1997); our results imply that this transfer-income restriction holds at most *locally* within a region, but fails to hold globally.

A paper closely related to ours is Chu (2020). Chu's focus is on the transfer-derivative restriction tested by AHK, (P3). She shows in detail how AHK's proof goes wrong by assuming continuity and differentiability and that the AHK transfer-derivative restriction only holds locally. We do not enter the details of AHK's proof, but our (constructive) results have the same implication. We go further than Chu's theory by characterizing how discontinuities propagate backward over the game's stages. Chu proposes an alternative local derivative test and calibrates a life-cycle model with two altruistic agents numerically.[8] Intriguingly, her results produce a transfer-income derivative far below one, and actually close to the one estimated by AHK. Chu thus concludes that AHK's test does not rule out altruism, opposed to what is claimed by AHK. In summary, our paper is more comprehensive on the theoretical side, yet less ambitious on the quantitative side; we thus view the two approaches as complementary.

A recent literature has embedded altruistic players into larger quantitative models. Kaplan (2012) studies the role of altruistic parents in insuring their children against labor-market risk by providing the possibility to move back home. In order to simplify strategic considerations, he assumes that parents cannot save. However, we find that the complications arising from strategic interactions remain even if the parent cannot save. Boar (2020) studies the importance of savings by altruistic parents to insure children against labor-income risk by providing inter-vivos transfers, focusing on equilibria with spoon-feeding transfers only. Both, Kaplan and Boar, rely on first-order conditions to solve the savings and transfer problems (i.e. the *local method*), which we show are inappropriate. Our results, also backed up quantitatively by the coarse calibration in Section 5, have the following implications for this quantitative literature: i) Welfare gains from insurance through altruistic transfers may well be larger than estimated in models solved by the local method.[9] ii) Locally-solved models may over-estimate the altruism parameters, since all gifts have to be generated by spoon-feeding to poor children. iii) Policy counterfactuals may yield erroneous results under the local method if the true response of households features an increase in the virtuous (autarkic) regime being played. iv) Numerical results in locally-solved models may be unreliable since discontinuities in value functions can induce large numerical value-function derivatives, feeding into volatile consumption and policy functions. Our paper contributes to this quantitative literature by guiding the quest for appropriate algorithms to solve this type of models.

[8] A caveat here is that Chu uses linear interpolation and quadrature methods, which we argued above may be problematic.

[9] Since back-loaded gifts induce the Samaritan's Dilemma, there is always an efficiency cost to gifts characterized by the FOC (1); but this is not the case for shots to autarky, thus increasing the welfare gains from altruistic insurance in a globally-solved model.

Finally, our results are of interest for a wider class of savings games. An important example is the problem faced by a hyperbolic discounter who plays "against" future versions of his self. The seminal paper in this literature is Harris & Laibson (2001); an important recent contribution is Cao & Werning (2018). Our paper shares with the latter that it uses non-local methods and allows for non-smoothness, yet a crucial difference is the infinite time horizon in their setting. How can our results inform such related literatures? First, our results on the propagation and multiplication of discontinuities and kinks suggest that the quest for infinite-horizon equilibria should entertain value and policy functions that have many –potentially infinitely many– discontinuities. Second, the chaotic nature of the propagation may entail that the finite-horizon equilibrium does not converge as the horizon approaches infinity; this matters since this limit is often used as an equilibrium-selection device. Finally, we have argued in previous work (Barczyk & Kredler, 2014*b* and Barczyk & Kredler, 2014*a*) that dynamic altruism models are more tractable in a) continuous time and b) adding noise to state variables. In the hyperbolic-discounting literature, argument a) is echoed by Cao & Werning (2016) and b) by Harris & Laibson (2003).

There is also a connection to one-player savings problems that display kinks in value functions and discontinuities in policies. Most closely related to our setting are consumption floors, i.e. means-tested government transfers as in Wellschmied (2021). For models with both continuous and discrete choices, Iskhakov et al. (2017) show how value-function kinks propagate back over time, value functions staying continuous and FOCs remaining necessary, however. We show that a two-player game takes non-smoothness one order higher, with value-function *discontinuities* that propagate backwards and FOCs being not even necessary.

The remaining paper is structured as follows. Section 2 provides the theoretical analysis of the benchmark model; most of the proofs are relegated to Appendix A. In Section 3, we study some applications of our framework, which are interesting in their own right. In Section 4 we extend the baseline model by (1) allowing the parent to save, and (2) by introducing uncertainty over the child's income. In Section 5, we contrast the erroneous local method with the correct global method qualitatively and provide a first approximation as to how far off the local method is quantitatively. Section 6 concludes.

2 An off-the-shelf model

We study a two-period deterministic model of a parent and a child. The two periods are denoted by $t = 0, 1$. In each period there are three stages. We refer to them as income (y), gift-giving (g), and savings (s) stage, respectively. In the first stage, the child receives income y_t^k and the parent receives income y_t^p. In the second stage, the parent decides on a non-negative transfer g_t to the child; the parent then consumes what is left and obtains utility from it. The third stage is the savings stage in which the child decides how much to save in a risk-free asset that pays a gross return $R > 0$; the child is subject to a no-borrowing constraint, i.e. we require $a_{t+1,y} \geq 0$.[10]

Preferences of the child are defined over the child's current- and future-period consumption (c_0^k, c_1^k) and represented by $u(c_0^k) + \beta u(c_1^k)$, where $\beta > 0$. The preferences of the parent include its own current- and future-period consumption (c_0^p, c_1^p) and also the consumption allocation of the child and are represented by $u(c_0^p) + \beta u(c_1^p) + \alpha[u(c_0^k) + \beta u(c_1^k)]$, where $\alpha > 0$ measures the strength of the parent's altruism towards the child. We make the following assumptions on the felicity function, which are standard and rather weak:

Assumption 1. *$u(\cdot)$ is twice continuously differentiable with $u'(c) > 0$ and $u''(c) < 0$ for all c and satisfies the Inada conditions $\lim_{c \to 0} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$.*

In order to study a non-trivial environment we will usually make use of the following condition:

Condition 1 (Gifts possible in final stage)**.** $\alpha u'(y_1^k) > u'(y_1^p)$.

It ensures that transfers are possible in the final period, which is the case if the parent wants to give to the child when the child has not saved anything. If this condition is violated the situation is entirely standard: Either i) autarky is the outcome or ii) the child receives transfers in $t = 0$, is borrowing-constrained, and receives no transfers in $t = 1$.

The state variable of the game is the child's *cash-on-hand* coming into each stage. Specifically, when entering the income stage at time t, we denote cash-on-hand by $a_{t,y}$. We will treat the assets $a_{0,y}$ with which the child enters the game as a parameter of our model. When entering the gift-giving stage, cash-on-hand is $a_{t,g} = a_{t,y} + y_t^k$. The parent takes $a_{t,g}$ as given and chooses child's cash-on-hand $a_{t,s} \geq a_{t,g}$, or expressed in gifts, $g_t = a_{t,s} - a_{t,g} \geq 0$. At the beginning of the savings stage the child's cash-on-hand is $a_{t,s} = a_{t,g} + g_t$.

To characterize the solution, we will make use of stage-contingent value functions. Let $V_{t,i}(a)$ be the child's value and $P_{t,i}(a)$ the parent's value when child's cash-on-hand is a coming into stage $i \in \{y, g, s\}$ of period $t \in \{0, 1\}$. In general, it is convenient to think of player's actions (gifts and savings) as setting cash-on-hand for the next stage of the game. We will denote the parent's cash-on-hand policy by $A_{t,s}(a_{t,g})$ and the child's consumption-savings policy by $A_{t+1,y}(a_{t,s})$.

We will illustrate our results with numerical examples. These are computed using the solutions that we derive for the power-utility case in Appendix B, which additionally invoke

Assumption 2 (Power utility)**.** *Utility is of the form $u(c) = (c^{1-\gamma} - 1)/(1 - \gamma)$, where $\gamma > 0$.*[11]

[10]Our results can easily be generalized to a more general borrowing limit $\underline{a} \neq 0$. However, it should be noted that setting a very low limit $\underline{a} < y_1^k/R$ enables the child to borrow against the parent's future endowment and thus to force altruistic transfers from the parent, which is clearly unrealistic.

[11]By a standard limit argument, and as is well-known, this implies that $u(c) = \ln c$ in the case $\gamma = 1$.

2.1 Final period

We solve the game by backward induction. Obviously, in the savings stage of the final period the child's optimal policy is to leave no resources behind. The policy and value functions are given by

$$A_{2,y}(a_{1,s}) = 0, \qquad V_{1,s}(a_{1,s}) = u(a_{1,s}), \qquad P_{1,s}(a_{1,s}) = \alpha u(a_{1,s}). \tag{2}$$

In order to keep track of the smoothness properties of value and policy functions, we state the following obvious result:

Lemma 1 (Smoothness in final stage)**.** *Under Ass. 1, value functions and the policy function in the final-period savings stage are twice continuously differentiable.*

Going back to the gift-giving stage of the final period, the parent's problem is then given by

$$P_{1,g}(a_{1,g}) = \max_{a_{1,s}\in[a_{1,g},a_{1,g}+y_1^p]} \big\{u(y_1^p + a_{1,g} - a_{1,s}) + P_{1,s}(a_{1,s})\big\}. \tag{3}$$

The parent chooses child's next-stage cash-on-hand, $a_{1,s}$. The lower bound of the feasible set, $a_{1,g}$, says that the parent must leave the child with at least as much as when entering the stage, which is nothing but the non-negativity constraint on gifts. The upper bound of the feasible set equals total family resources, $a_{1,g} + y_1^p$, i.e. the maximal transfer the parent can give if its income is y_1^p. Combining Lemma 1 and Ass. 1, we see that the maximization problem (3) is well-behaved, i.e. the parent maximizes a concave criterion on a convex set.

In general, our strategy for solving the game will be as follows. We define well-behaved auxiliary problems in each stage that describe the solution in a particular regime; for example, a regime in the final-period gift-giving stage is if gifts flow or not. We then characterize regime-specific value and policy functions and piece them together to find the global solution to the game.

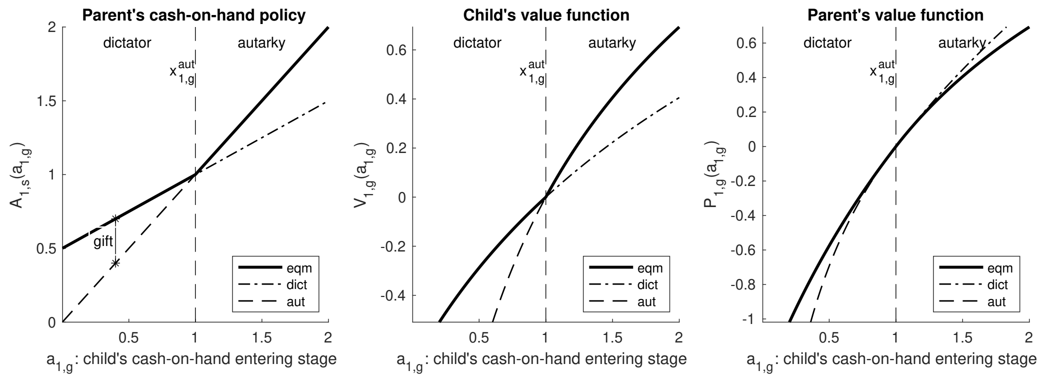
In the final-period gift-giving stage, the first regime we define is the *dictator* (dict) environment in which we give the parent the power over all the family's resources, i.e. we allow the parent to choose positive as well as negative gifts:

$$P_{1,g}^{dict}(a_{1,g}) = \max_{a_{1,s}\in[0,a_{1,g}+y_1^p]} \left\{u(y_1^p + a_{1,g} - a_{1,s}) + P_{1,s}(a_{1,s})\right\}. \tag{4}$$

This problem differs from the true problem (3) only in that it enlarges the feasible set. It is easy to see that concavity and the Inada condition from Ass. (1) guarantee a unique interior solution to the dictator problem. This solution is implicitly defined from the parent's first-order condition

$$u'\big(\underbrace{y_1^p + a_{1,g} - A_{1,s}^{dict}(a_{1,g})}_{=c_1^p}\big) = \alpha u'\big(\underbrace{A_{1,s}^{dict}(a_{1,g})}_{=c_1^k}\big). \tag{5}$$

Figure 1: Gift-giving stage at $t = 1$



Equilibrium and auxiliary outcomes in gift-giving stage of the final period. Baseline parameters: $u(c) = \ln(c)$, $\alpha = \beta = R = 1$, $y_0^p = y_1^p = 1$, and $y_1^k = 1/4$.

We recognize the familiar first-order condition for altruistic transfers from (1), which says that the parent equates marginal utility from own consumption to the marginal utility from child consumption (weighted by the strength of altruism). Equation (5) holding is what is commonly referred to as an *operative* transfer motive.

The second auxiliary problem we define in the final-period gift-giving stage is *autarky* (aut), i.e. an environment in which we force gifts to be zero. The policy and value functions in this environment are given by

$$A_{1,s}^{aut}(a_{1,g}) = a_{1,g}, \qquad V_{1,s}^{aut}(a_{1,g}) = u(a_{1,g}), \qquad P_{1,s}^{aut}(a_{1,g}) = u(y_1^p) + \alpha u(a_{1,g}). \tag{6}$$

The solution to the parent's actual problem (3) is now straightforward.[12] The optimal gift policy is given by the maximum of the two auxiliary policies since the child must have at least what it has under autarky:

$$A_{1,s}(a_{1,g}) = \max\left\{A_{1,s}^{dict}(a_{1,g}), A_{1,s}^{aut}(a_{1,g})\right\} = \max\left\{A_{1,s}^{dict}(a_{1,g}), a_{1,g}\right\}. \tag{7}$$

The parent chooses a positive gift when the child is poor, but then switches to autarky once the child is rich enough (the parent would want to take away from the child, but cannot). We can characterize the threshold $x_{1,g}^{aut}$ at which the regime changes from dictator to autarky by

$$u'(y_1^p) = \alpha u'(x_{1,g}^{aut}), \tag{8}$$

which we note to be always well-defined under Ass. 1. Note that Cond. 1 implies $x_{1,g}^{aut} > y_1^k$, which

[12] We discuss the problem here somewhat informally, but provide a formal statement and proof in Lemma 2.

means that the child will receive gifts if her savings are low enough.

Fig. 1 shows the policy (7) and the associated value functions for an example with logarithmic utility; since utility is homothetic, the parent always assigns the same share of resources to the child as a dictator and the optimal policy is thus piecewise linear. It is worthwhile to observe what the regime change implies for the smoothness of the value functions. Since the parent equates the marginal utility from consuming and from giving the first dollar in gifts at $x_{1,g}^{aut}$, the parent's value function is differentiable (by the Envelope Theorem). The child, however, cares only about her own consumption. Hence the kink in the parent's gift-giving policy (in the left panel) directly translates into a kink in the child's value function (in the middle panel).

In the following Lemma we summarize the most important features of the gift-giving stage in the final period (for the proof see Appendix A). Note here that these features hold for a much larger class of felicity functionals than the logarithmic one in our example.

Lemma 2 (Final period: kinks in gift-giving stage)**.** *Suppose that Ass. 1 holds and let $x_{1,g}^{aut} > 0$ be defined by Eq.* (8)*. Then the parent's policy at $t = 1$ is to give positive gifts for $a_{1,g} < x_{1,g}^{aut}$, but no gifts for $a_{1,g} \geq x_{1,g}^{aut}$. The parent's policy function $A_{1,s}(a_{1,g})$ is continuously differentiable everywhere except for the point $x_{1,g}^{aut}$, satisfying*

$$A'_{1,s}(a_{1,g}) \begin{cases} \in (0,1) & \textit{for } a_{1,g} < x_{1,g}^{aut}, \\ = 1 & \textit{for } a_{1,g} > x_{1,g}^{aut}, \end{cases}, \tag{9}$$

$$A'^{-}_{1,s}(x_{1,g}^{aut}) < A'^{+}_{1,s}(x_{1,g}^{aut}) = 1, \tag{10}$$

i.e. there is a downward kink in the policy function between the two regions.[13] *The child's value function $V_{1,g}(\cdot)$ has the same smoothness profile as the policy function, i.e. it is continuously differentiable except for a downward kink at $x_{1,g}^{aut}$. However, the parent's value function $P_{1,g}(\cdot)$ is continuously differentiable everywhere with*

$$P'_{1,g}(a_{1,g}) = \alpha u'\big(A_{1,s}(a_{1,g})\big). \tag{11}$$

The parent's value function $P_{1,g}(\cdot)$ is (globally) strictly concave; the child's value function $V_{1,g}(\cdot)$ is strictly concave on the range $(x_{1,g}^{aut}, \infty)$.

Three remarks are in order. First, the parent's marginal propensity to give in (9) is smaller than unity on the range where gifts are positive; this is the "tax" that the parent applies on the child's savings that can give rise to the Samaritan's Dilemma. Second, Cond. 1 is not required for the characterization in this Lemma. However, if Cond. 1 does not hold, the dictator/gift-giving region

[13] We denote by $f'^{-}(x)$ the left and by $f'^{+}(x)$ the right derivative of $f(\cdot)$ at x.

$a_{1,g} < x_{1,g}^{aut}$ cannot be reached on the equilibrium path. Third, in the example in Fig. 1 the child's value function is concave on the range $(0, x_{1,g}^{aut})$ because of the linear gift-giving policy. This, however, need not be the case in general since gift-giving policies may be convex for different utility specifications.

2.2 Initial period

2.2.1 Savings stage

In the initial period, the child enters the savings stage with cash-on-hand $a_{0,s}$. Given these resources, the child chooses period-1 cash-on-hand $a_{1,y}$, subject to a no-borrowing constraint, by solving the problem

$$V_{0,s}(a_{0,s}) = \max_{a_{1,y} \in [0, Ra_{0,s}]} J(a_{1,y}; a_{0,s}) \tag{12}$$

where the child's *criterion function* is given by

$$J(a_{1,y}; a_{0,s}) = u\big(a_{0,s} - a_{1,y}/R\big) + \beta V_{1,y}(a_{1,y}). \tag{13}$$

We note that the child's continuation value when entering the period-1 income stage is given by $V_{1,y}(a_{1,y}) = V_{1,g}(a_{1,y} + y_1^k)$.

It turns out that the child's problem in (12) is non-standard since the criterion function $J(\cdot; a_{0,s})$ is non-concave, which is directly implied by the downward kink of $V_{1,y}(\cdot)$. Figure 2 shows examples of the child's payoff function, fixing different values of cash-on-hand $a_{0,s}$ and varying savings choices $a_{1,y} \in [0, Ra_{0,s}]$. The critical feature in the figure is the downward kink in the child's criterion function:

Lemma 3 (Properties of child's criterion function $J(\cdot; a_{0,s})$ in savings stage)**.** *Suppose that Ass. 1 holds and define $x_{1,y}^{aut} = x_{1,g}^{aut} - y_1^k$ as the minimal savings that make the child autarkic at $t = 1$. For fixed $a_{0,s}$, the function $J(a'; a_{0,s})$ defined in Eq.* (13) *is continuous in a', concave in a' for $a' \geq x_{1,y}^{aut}$ and differentiable everywhere except in the point $a' = x_{1,y}^{aut}$. If Cond. 1 holds, then $J(\cdot; a_{0,s})$ has a downward kink when entering autarky, i.e. $J'^{-}(x_{1,y}^{aut}; a_{0,s}) < J'^{+}(x_{1,y}^{aut}; a_{0,s})$.*

Proof. Given the definition of $J(\cdot)$ in Eq. (13), the claimed properties follow directly from Ass. 1 and the properties of $V_{1,y}(a) = V_{1,g}(a + y_1^k)$ from Lemma 2. ■

We now characterize the optimal savings. As is well-understood, in the dictator regime the parent decreases ("taxes") gifts as the child saves more. However, this wedge is not present in the autarkic regime. This becomes visible when taking the first-order condition in (12) that gives us the child's Euler equation:

$$u'(c_0^k) \geq R\beta u'(c_1^k) A'_{1,s}(a_{1,g}), \tag{14}$$

where $a_{1,g} = a_{1,y} + y_1^k$ and which must hold with equality whenever savings are positive. From the properties of $J(\cdot)$, it is clear that this equation is necessary but not sufficient for a solution; furthermore, it holds with inequality when the child is borrowing-constrained. In the Euler equation (14), observe that for savings $a_{1,y}$ such that the child is autarkic, i.e. $a_{1,g} > x_{1,g}^{aut}$, we have $A'_{1,s}(a_{1,g}) = 1$ and thus the standard Euler equation obtains. For savings below the threshold $x_{1,g}^{aut}$, the parent responds by reducing the transfer to the child in the final period, $0 < A'_{1,s}(a_{1,g}) < 1$, which leads to an Euler equation with a distortion. This wedge creates a disincentive for the child to save.

In order to state our main results for the savings stage, we first establish that the child's savings policy is weakly increasing.

Lemma 4 (Increasingness of savings correspondence)**.** *Under Ass. 1, the savings correspondence $A_{1,y}(a)$ is increasing in the following sense: If savings A are optimal for some state a, then an optimal savings policy for any higher starting wealth $a + \delta$ must be such that at least A is saved. To be precise, if $A \in A_{1,y}(a)$ for fixed a, then $A - \epsilon \notin A_{1,y}(a + \delta)$ for any $\epsilon \in (0, A]$, for all $\delta > 0$.*

We now have everything in place to characterize the savings stage in the initial period.

Proposition 2.1 (Discontinuous policy and value function in savings stage)**.** *Suppose that Ass. 1 and Cond. 1 hold. Then there exists $x_{0,s}^{aut} \in (0, \infty)$ such that the child chooses savings leading into autarky at $t = 1$ for all $a_{0,s} > x_{0,s}^{aut}$, while savings are such that gifts flow at $t = 1$ for $a_{0,s} < x_{0,s}^{aut}$. Optimal savings on the autarky range are characterized by a continuous function. The savings correspondence $A_{1,y}(\cdot)$ is discontinuous (with an upward jump) at $x_{0,s}^{aut}$. The child's value function $V_{0,s}(\cdot)$ is continuous at $x_{0,s}^{aut}$, but the parent's value function $P_{0,s}(\cdot)$ has an upward jump discontinuity at this threshold.*[14]

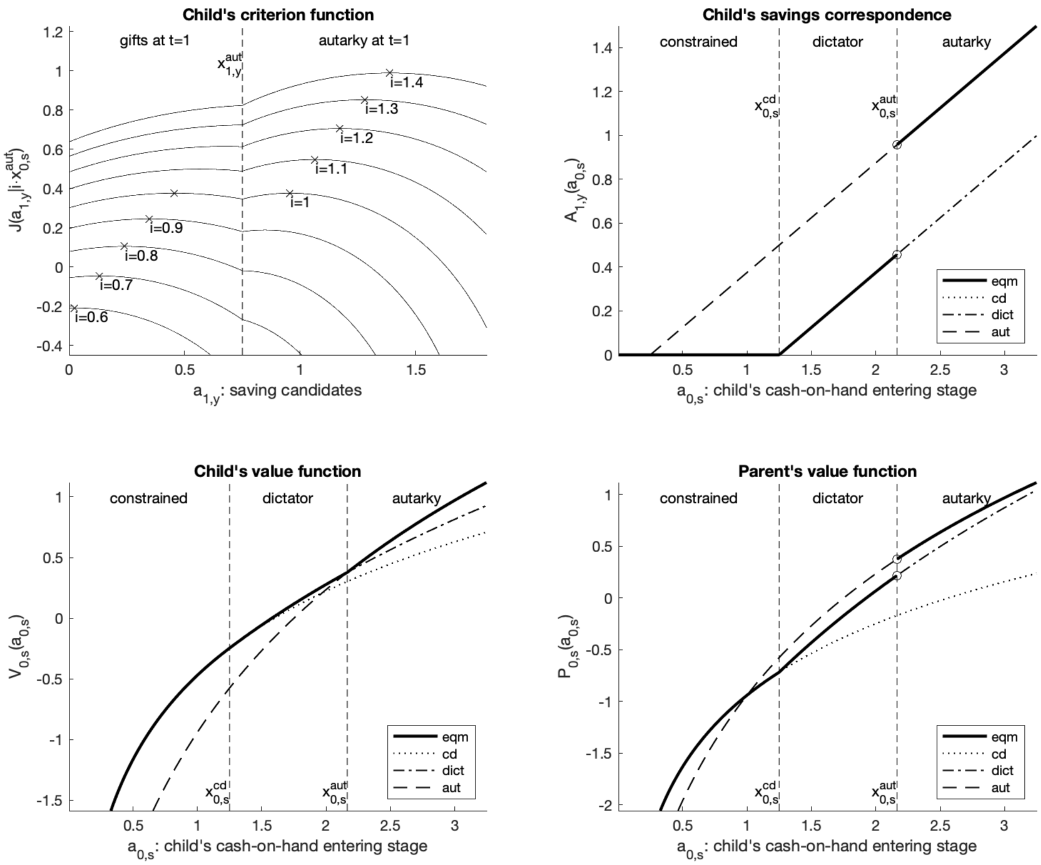
It is worthwhile to point out that the proof for this proposition, which is provided in the appendix, invokes the Inada condition $\lim_{c \to \infty} u'(c) = 0$, which implies that the child chooses autarky for high-enough starting wealth. If marginal utility did not vanish (e.g. linear utility), then the child may always prefer to consume today in order to maximize transfers in the final period.

Fig. 2 demonstrates Prop. 2.1.

Discontinuous policy: In the upper left panel, we can see that for intermediate values of child's cash-on-hand, $a_{0,s}$, the child's payoff function has two local maxima: the first optimum is such

[14] If Cond. 1 does not hold, then the proposition's claims still hold when setting $x_{0,s}^{aut} = -\infty$, since any savings policy by the child leads to autarky at $t = 1$.

Figure 2: Savings stage at $t = 0$



Equilibrium and auxiliary outcomes in savings stage of the initial period. (a) Child's payoff J as a function of next-period savings, $a_{1,y}$, for selected values of current-period cash-on-hand, $a_{0,s} = i \cdot x^{aut}_{0,s}$. Baseline parameters: $u(c) = \ln(c)$, $\alpha = \beta = R = 1$, $y^p_0 = y^p_1 = 1$, and $y^k_1 = 1/4$.

that gifts flow at $t = 1$ (*dictator savings*), and the second one corresponds to *autarkic savings*. As is intuitive, the child selects dictator savings when poor but switches to the local optimum corresponding to autarkic savings when richer, see the second graph in the upper panel. There must be some level of cash-on-hand at which the child is just indifferent between dictator and autarkic savings, i.e. the payoff function has two global maxima, which is denoted in the proposition by $x_{0,s}^{aut}$. Since autarkic savings are discretely larger than dictator savings due to the downward kink in the criterion, the child's savings policy is discontinuous at $x_{0,s}^{aut}$.

Value functions: The lower panel shows the child's (the first graph) and the parent's (the second graph) value function in this stage of the game. While the child's value function is continuous at $a_{0,s} = x_{0,s}^{aut}$, the child being indifferent between the two regimes, this is not the case for the parent. In fact, the parent strictly prefers the autarkic savings allocation at $x_{0,s}^{aut}$, which manifests itself in a discrete upward jump of the parent's value function. The part of the parent payoff that stems from the child's consumption is continuous at this point, since the child is indifferent. However, the part of the parent payoff that stems from the parent's own consumption is strictly higher in autarky since the parent consumes strictly more in autarky, which the child does not take into account in her decision.

Finally, we observe that the no-borrowing constraint, when binding, adds an additional kink to the savings policy and to the parent value function. This occurs at $x_{0,s}^{cd}$, which we define as the maximal level of wealth below which the child is constrained. Similar to the final-period gift-giving stage analyzed before, the child's value function is differentiable at $x_{0,s}^{cd}$ by the Envelope Theorem, but the parent's value function has a kink. The reason is that the parent has an additional advantage from child savings – the fact that the parent has to provide lower gifts and can thus consume more at $t = 1$, an externality that the child ignores in its decision.

2.2.2 Gift-giving stage

Prior to entering the savings stage, the parent has the possibility to manipulate the child's level of cash-on-hand in the gift-giving stage and thereby to influence the child's ensuing savings choice. Specifically, in this stage the parent takes the child's level of cash-on-hand, $a_{0,g}$, as given and considers whether stocking it up would improve her well-being, solving

$$P_{0,g}(a_{0,g}) = \max_{a_{0,s} \in [a_{0,g}, a_{0,g} + y_0^p]} K(a_{0,s}; a_{0,g}),$$

where the parent's criterion function is given by

$$K(a_{0,s}; a_{0,g}) = u(y_0^p + \underbrace{a_{0,g} - a_{0,s}}_{=-g_0}) + P_{0,s}(a_{0,s}). \tag{15}$$

Just as is the case in the final period, the parent chooses child's next-stage cash-on-hand subject to not being able to extract resources from the child. As was the case for the child's savings decision, the parent's decision problem is non-standard but to an even starker extent: In addition to a kink, the parent also faces a discontinuity in the criterion $K(\cdot; a_{0,g})$, which is inherited from the continuation value $P_{0,s}(\cdot)$. Thus, first-order conditions are no longer even necessary for a global optimum. The upper left panel of Fig. 3 plots $K(\cdot; a_{0,g})$ for fixed levels of $a_{0,g}$.[15] We clearly see the upward jump when the autarky regime is entered and the kink at the threshold where the child switches from being constrained to dictator-savings.

Before we describe how we find the optimal cash-on-hand policy for the parent, it is again useful to first establish its monotonicity:

Lemma 5 (Increasing cash-on-hand correspondence at $t = 0$)**.** *Under Ass. 1, the optimal cash-on-hand correspondence $A_{0,s}(a)$ is increasing in the sense of Lemma 4: If A is optimal for a given state a, then the parent will not choose gifts below A for higher states. To be precise, if $A \in A_{0,s}(a)$ for fixed a, then $A - \epsilon \notin A_{0,s}(a + \delta)$ for any $\epsilon > 0$, for all $\delta > 0$.*

A direct corollary of this lemma is that the sequencing of regimes is the same as in the child-savings stage:

Corollary 2.1 (Sequence of regimes in gift-giving stage at $t = 0$)**.** *The sequence of regimes in the first-period gift-giving stage is (i) constrained, (ii) dictator-savings and (iii) autarky, where (i) or (ii) or both may be skipped. Specifically, there exist numbers $x_{0,g}^{aut} \geq x_{0,g}^{cd} \geq 0$ such that (i) for states $a_{0,g} < x_{0,g}^{cd}$ the equilibrium is such that the child will be constrained, (ii) for states $x_{0,g}^{cd} < a_{0,g} < x_{0,g}^{aut}$ the child chooses positive savings but gifts flow in the final period (dictator savings), and (iii) for states $a_{0,g} > x_{0,g}^{aut}$ the child is in autarky in the final period.*

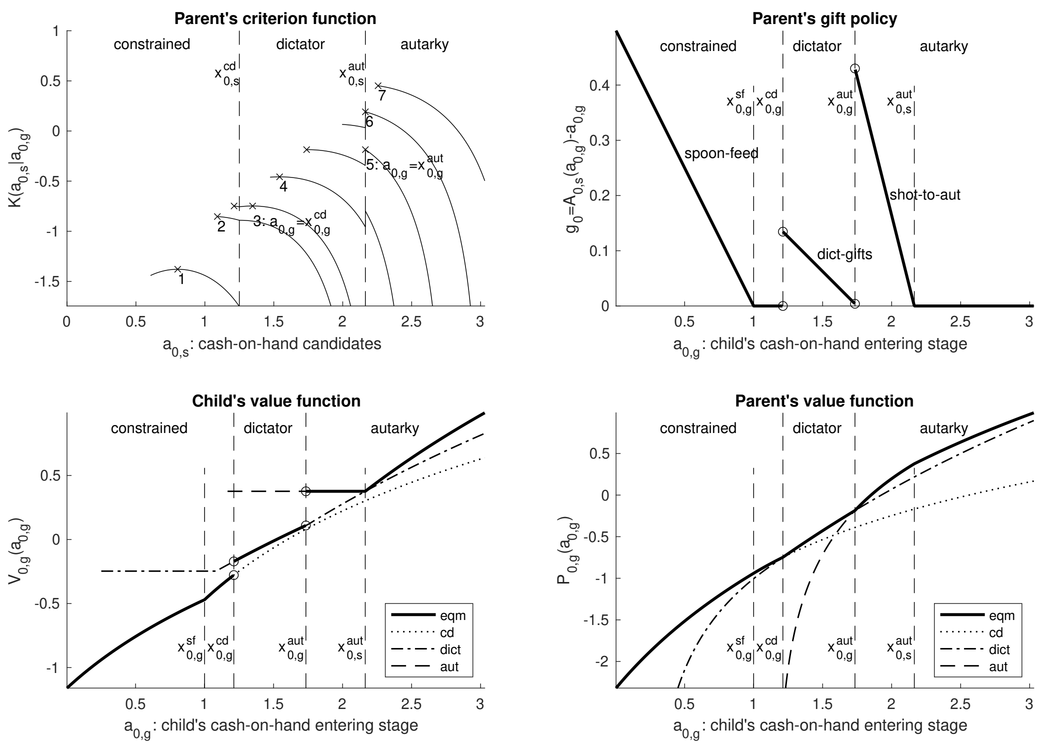
Remark: On the boundaries between two regimes, the parent is indifferent between the adjacent regimes and either policy is compatible with equilibrium.

Our strategy to find the optimal cash-on-hand policy is now the following: i) Find the local maximum within each regime, which can (usually) be done using first-order conditions.[16] ii) Find the thresholds at which the parent is indifferent between the local maxima of neighboring regimes (value-matching), which give us the cut-offs $x_{0,g}^{cd}$ and $x_{0,g}^{aut}$ in Corollary 2.1.

[15] To generate Fig. 3, we have chosen parameters that lead to a large number of regimes to allow for a comprehensive discussion; in Section 3 we explore a parameter configuration under which regimes disappear, which is interesting in its own right.

[16] FOCs can *always* be used for the constrained and autarkic regime. For the dictator-savings region, we can show that under power utility K is differentiable and concave. For other felicity functions $u(\cdot)$, however, K may be non-concave or even discontinuous in the dictator-savings regime if the child's savings policy is convex or discontinuous on this range.

Figure 3: Gift-giving stage at $t = 0$



Equilibrium and auxiliary outcomes in gift-giving stage of the initial period. Upper left panel shows parent's payoff K as a function of child's next-stage cash-on-hand $a_{0,s}$ for selected (fixed) values of current-stage cash-on-hand $a_{0,g}$. Value functions for dict and aut shown on range where respective outcome is feasible. Baseline parameters: $u(c) = \ln(c)$, $\alpha = \beta = R = 1$, $y_0^p = y_1^p = 1$, and $y_1^k = 1/4$.

For step (i), we need to calculate the derivative of the criterion, $K'(a_{0,s}; a_{0,g}) = -u'(y_0^p + a_{0,g} - a_{0,s}) + P'_{0,s}(a_{0,s})$, on its smooth parts. Particular care has to be taken when evaluating the parent's marginal continuation value, which we do in Prop. A.1 in the appendix. It turns out that only in the constrained and autarkic regions, the conjecture of previous literature holds and we have $P'_{0,s} = \alpha u'(c_0^k)$, which then leads to the static gift FOC 1 (for interior local optima). In the dictator region, we show that there is an extra term that captures an additional marginal benefit of gift-giving: Higher initial gifts entail higher child savings, which in turn translate into lower final-period gifts and thus higher parent consumption in the future.

We now return to Fig. 3 to explain the parent's optimal gift policy.

Constrained regime. Let us first consider very low levels of child cash-on-hand $a_{0,g}$. For these, the optimal gift will be such that the child is constrained and consumes all of the gift, which corresponds to lines 1-2 in the upper left panel. Two cases are possible here: An interior optimum can occur, as is the case for line 1. Gifts are then positive, which corresponds to the region left of $x_{0,g}^{sf}$ in the upper-right panel. This is the typical *spoon-feeding* gift that the literature has focused on:

The child consumes hand-to-mouth and thus the parent effectively controls the child's consumption. At some point of the constrained regime, however, the non-negativity constraint on gifts can bind (as in criterion 2 in the upper left panel) and gifts become zero, as is the case on the interval $a_{0,g} \in (x^{sf}_{0,g}, x^{cd}_{0,g})$ in the upper-right panel. In the lower two panels, we see that at the switch from positive to zero gifts within the constrained region the parent's value function is smooth (due to the Envelope Theorem), whereas a new downward kink is introduced into the child's value function, as is familiar by now.

Dictator regime. At the point $x^{cd}_{0,g}$, the parent is then indifferent between the best option in the constrained and dictator regimes, which is the case for criterion 3 in the upper-left panel. The parent then switches to a positive gift that takes the child into the dictator-savings regime, corresponding to the first spike of the gift function at $x^{cd}_{0,g}$. Since the parent is indifferent, her value $P_{0,g}$ is continuous (but has a kink) at $x^{cd}_{0,g}$. The child's value function, however, has an upward jump since the child strictly prefers the higher gift. When increasing $a_{0,g}$ further above $x^{cd}_{0,g}$, the parent decreases gifts. There can be another kink in the gift-giving function within the dictator-savings regime if the non-negativity constraint on gifts binds, but this is not the case for the parameters chosen in Fig. 3.[17]

Autarky regime. Finally, at the threshold $x^{aut}_{0,g}$ the parent is indifferent between the best choice on the dictator-savings range and shooting the child to the autarky threshold, corresponding to criterion 5 in the upper left panel and the second spike of the gift function. Since the parent is indifferent at the threshold, the parent's value function is continuous (with a kink) at $x^{aut}_{0,g}$. The child, again, strictly prefers the larger gift and there is another upward jump in her value function at this threshold. For child cash-on-hand above $x^{aut}_{0,g}$, the parent's optimal policy is then to shoot the child to autarky, at least as long as this is necessary: Line 6 in the upper left panel depicts another shot to autarky, whereas for line 7 the shot is not necessary and the optimal gift is zero, corresponding to the area right of $x^{aut}_{0,s}$. The child value function is flat on the range $(x^{aut}_{0,g}, x^{aut}_{0,s})$ since all situations lead to the same outcome for the child; then there is a kink at $x^{aut}_{0,s}$ once gifts become zero.

One may now ask how stable a feature the discontinuities in the gift function are. It turns out that the second spike in the gift policy, shots to autarky, are very robust (since the discontinuity in the parent's continuation value is). The following is one of the main results of the paper:

Proposition 2.2 (Shots to autarky)**.** *Under Ass. 1 and Cond. 1, there is a range of initial states,* $a_{0,g} \in (x^{aut}_{0,s} - \epsilon, x^{aut}_{0,s})$ *with* $\epsilon > 0$*, for which the equilibrium is such that the parent gives a gift at* $t = 0$ *and the child is in autarky at* $t = 1$.

Proof. Fix $\epsilon > 0$ small and consider the range of states $a_{0,g} \in (x^{aut}_{0,s} - \epsilon, x^{aut}_{0,s})$. The payoff

[17] When decreasing curvature to γ to 0.8, for example, this kink is present. We chose log-utility for the figure since the results are easier to interpret.

of shooting the child to autarky can be lower-bounded for all $a_{0,g}$ on this range by $L^{aut}(\epsilon) = P^{+}_{0,s}(x^{aut}_{0,s}) + u(y^p_0 - \epsilon)$, since at most ϵ must be given for any $a_{0,s}$ to reach autarky.[18] Similarly, the payoff of any policy that maintains the child in the dictator-savings regime can be upper bounded by $U^{dict}(\epsilon) = P^{-}_{0,s}(x^{aut}_{0,s})+u(y^p_0)$.[19] As we let $\epsilon \to 0$, we obtain $L^{aut}(\epsilon) > U^{dict}(\epsilon)$ since $u(y^p_0-\epsilon) \to u(y^p_0)$ but the function $P_{0,s}$ has an upward jump, i.e. $P^{+}_{0,s}(x^{aut}_{0,s}) - P^{-}_{0,s}(x^{aut}_{0,s}) = \delta > 0$ by Prop. 2.1. Thus, the parent's optimal policy must be a gift that makes the child reach the autarky region. ■

In the example of Fig. 3, all shots to autarky are *point-landings*: The parent provides just enough so that the child stays autarkic. But, can there be situations in which the parent chooses *interior solutions* within the autarky region? Prop. A.2 in the appendix states that the sufficient condition for this to occur is a high interest rate. Thus, there are cases in which the child receives transfers even though the child would have chosen autarkic savings anyway, which may seem counterintuitive at first. In Section 3.1 we will study such a case.

2.3 Proliferation of non-smoothness and conjectures for longer horizons

Finally, it is worthwhile to summarize the pattern that has emerged in the multiplication of regimes, kinks and jump discontinuities. Table 1 provides an overview for the case of power utility. The final-period consumption stage is the only fully well-behaved stage, featuring smooth value and policy functions. The final-period gift-giving stage is the only of the remaining stages in which the maximization problem is well-behaved in the sense that the FOC is both necessary and sufficient. The first kink in the policy function shows up here, induced by the non-negativity constraint on gifts. This introduces a kink in the child's value function, thus the FOC is no longer sufficient in the child's savings problem in the stage before (yet it is still necessary). The first discontinuities in the policy and parent value functions also show up in this stage at the point where the child switches from the dictator (local) maximum to the autarkic (local) maximum. Furthermore, an additional kink can appear in this stage, induced by the child's borrowing constraint. Finally, in the initial gift-giving stage, the parent's maximization problem is very ill-behaved, FOCs being neither necessary nor sufficient, and multiple kinks and discontinuities are possible, as shown in Figure 3.

We now analyze the general pattern in broad brushes in an effort to provide conjectures about the properties of an extended multi-period model. Within any smooth regime in a given stage, the non-negativity constraint on the first player's policy (gifts or savings) can introduce a downward kink (i.e. a strong convexity) in the policy. This leads to a downward kink in the second player's

[18] $P^{+}_{0,s}(x)$ denotes the right limit of $P_{0,s}$ at x here. We assume in this proof that the child chooses autarky at $x^{aut}_{0,s}$ when indifferent, which simplifies the exposition. The argument has to be modified slightly if dictator-savings is selected as the equilibrium policy at $x^{aut}_{0,s}$.

[19] Here, $P^{-}_{0,s}(x)$ denotes the left limit at x.

Table 1: Proliferation of kinks and discontinuities under power utility

Stage	$t=1$ consumption	$t=1$ gift-giving	$t=0$ savings	$t=0$ gift-giving
max. # smooth intervals	1	2	3	7
FOC necessary	NA	yes	yes	no
FOC sufficient	NA	yes	no	no
max. # policy jumps	0	0	1	2
max. # policy kinks	0	1	1	4
max. # parent-value jumps	0	0	1	0
max. # child-value jumps	0	0	0	2

smooth intervals: Intervals in state space within which value and policy functions are continuously differentiable. *FOC*: first-order condition. *NA*: not applicable. *Jumps*: Jump discontinuities. *Kinks*: Points where function is continuous, but first derivative is not. *Notes:* i) Results in this table follow from Lemmas and Propositions in Section 2 and Appendix A. ii) # policy kinks in $t = 0$ gift-giving stage: There can be up to two kinks within the autarky regime, but at most one kink within each the constrained and the dictator regime since right corner solutions are impossible in the latter two cases. iii) Table is for case of power/logarithmic felicity; for general felicity functions there may be additional discontinuities in the child's savings policy within the dictator regime if Cond. (A.1) fails; these discontinuities then have to be taken into account in the $t = 0$ gift-giving stage.

value function, whereas the first player's value function has a continuous first derivative at this point by the Envelope Theorem. This is the crucial point why non-smoothness escalates in this two-player game but not in a one-player savings problem. Going back one stage, this downward kink in the second player's value function can then induce an upward jump in the second player's policy when the global maximum jumps from one regime to the next. This jump, in turn, leads to an upward jump in the first player's value function. The second player's value function, however, is continuous at this point since she must be indifferent between the two regimes.[20] This pattern leads us to the following **conjectures for multi-period models**:

C1 The decision maker's value functions in a given stage are continuous, but have kinks, while the other player's value functions display discontinuities (and also kinks), whose number increases going backward in time. Both players' value functions are non-decreasing in all stages.

C2 The potential number of smooth intervals strictly increases as we go backward in time. The actual number of intervals can increase, stay constant or decrease, depending on parameters.[21] The ordering of these intervals in the state space stays the same from one stage to the next.

C3 FOCs are informative to find local maxima within a smooth interval, but uninformative to

[20] Regimes that are characterized by value-function jumps at their boundary can even split up into three new regimes, as is the case for the autarkic regime in the initial gift-giving stage, where the parent policy can be a left corner solution (shot to autarky), an interior solution (shot *into* autarky), or a right corner solution (zero gifts).

[21] As the example of the college case in Section 3.1 will show, a regime may disappear: In the example, the dictator regime with gifts in the final period is not reached on the equilibrium path.

identify global maxima. Global maxima have to be determined by explicitly comparing the values from the different local maxima.

Despite this complexity there may be a silver lining when taking the game to an infinite horizon if the game converges to a limit, but this is highly speculative. Such a limit may well feature infinitely many smooth intervals. Finally, it is worth noting that our analysis also offers lessons for general games with continuous state variables and continuous controls, especially if they feature occasionally-binding constraints on policies (gifts and savings, in our case).

3 Applications

We now explore some parameter configurations of our setting and shows that this simple model, solved properly, can give rise to a set of predictions that have not been identified by previous literature. This may open up new applications of the altruism model. We first study an extreme parameter configuration under which the parent wants to save through the child – the "college case". Second, we exploit closed-form solutions for the power-utility case to shed light on how the presence of an altruistic donor shapes savings behavior. We focus on the marginal propensity to save, a key object in macroeconomic policy analysis.

3.1 College case

It turns out that for certain parameter configurations, the equilibrium of our model is such that i) gifts flow only in the initial period, i.e. the game always ends in the autarky regime and ii) the parent provides initial gifts that go *into* the autarky region and not just to its threshold, i.e. the parent effectively saves through (or invests in) the child. This extreme case highlights that the properly-solved model can lead to predictions that are diametrically opposed to the Samaritan's Dilemma that previous literature had emphasized: Child savings are always efficient and all transfers are front-loaded. We will now argue that parents' investments in child education could be a candidate for this type of equilibrium. To find parameters that lead to i) and ii), we make use of Prop. A.2 which suggests to set a high interest rate R. Figure C.1 in the appendix shows an example with $R = 3$, the remaining parameters being as under the baseline. For the savings-through-the-child equilibrium to arise, one assumption is crucial: The parent does not have access to the savings technology with the high return R. If she had, the parent would save herself instead of giving to the child, this being a more effective way of increasing the parent's future consumption than through child savings. Thus, potential empirical applications of this type of equilibrium have to involve an asset that i) has a high return and ii) only the child has access to. A child's college education may be just this.

3.2 Changing the inter-temporal elasticity of substitution

All our previous examples have made use of logarithmic utility, but we derive solutions for the more general case of power utility in Appendix B. In general, the power-utility coefficient affects the slope of policy functions and the configuration of regimes. We highlight here one novel result on the child's savings behavior, which seems puzzling at first. We show that for the empirically relevant case of $\gamma > 1$ (i.e. an inter-temporal elasticity of substitution below unity), the child's marginal propensity to save is *higher* when expecting future gifts (i.e. in the dictator regime) than in autarky; see Prop. B.1 in the appendix for the formal statement. At first sight, this seems at odds with the Samaritan's Dilemma, which says that future gifts disincentivize savings. However, the result is indeed consistent with it, since the child's savings are lower in *levels* in the dictator regime, whereas their *slope* is higher (which is what the marginal propensity to save is about). The key insight to understand this result is that an altruistic donor effectively lowers the interest rate by taxing savings. At a lower interest rate, a saver has to assign a higher fraction of additional income to savings so as to keep a smooth consumption path, which is paramount for a saver with low intertemporal elasticity (high curvature in utility).

4 Extending the baseline model

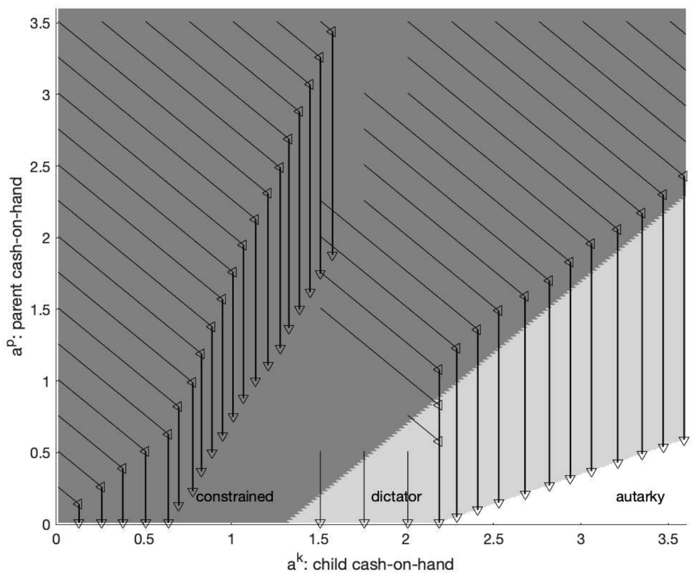
In this section we show through a series of extensions that the key results obtained from our baseline model are robust.

4.1 Parent can save

So far we have left out the parent's savings decision, which was obviously not for realism but for tractability. We will now show that by doing this, we have not lost anything of qualitative significance, i.e. the same strategic considerations remain intact when the parent can save. This is crucial since it stands in contrast to literature which has argued that disallowing savings by one of the two agents circumvents complications arising from strategic considerations.

For this subsection we consider the following modification to the baseline setting. In the initial period, the parent first chooses gifts and savings (at the same interest rate R as the child and subject to a no-borrowing constraint), and, second, the child chooses savings. The final period is identical to before. The child's payoff-relevant state in the child-savings stage is now given by $(a^k_{0,s}, a^p_{1,y})$, where $a^p_{1,y}$ is the parent's savings choice made in the first stage. First, it turns out that we can recycle one of our previous results for the child-savings stage, since we can treat the parent's (fixed) savings as a part of her final-period endowment:

Figure 4: Parent can save (outcomes at $t = 0$)



Dynamics induced by parent's decision in initial period when parent can save. Diagonal vectors show displacements in state space due to first-period gifts. Note here that any $(a^k_{0,g}, a^p_{0,g})$ that falls on a diagonal line is moved to the tip of the vector in the gift-giving stage; this is a common property of altruism models. Vertical vectors show movements due to parent's savings choices, conditional on the parent having provided a positive gift (to avoid cluttering the diagram); we also show selected arrows that end in the dictator regime. Shaded areas correspond to regimes in ensuing child-savings stage, marking if the child saves zero ("constrained"), saves such that gifts occur at $t = 1$ (dictator) or saves such that no gifts occur at $t = 1$ ("autarky"). Parameters: $u(c) = \ln(c)$, $\alpha = \beta = R = 1$, $y^p_0 = y^p_1 = 1$, and $y^k_1 = 1/4$.

Corollary 4.1 (Discontinuities when parent can save)**.** *Consider an alternative environment in which the parent can also allot resources to savings (besides gifts and consumption). In the initial period's child-savings stage, the policy function $A_{1,y}(\cdot)$ and the parent value function $P_{0,s}(\cdot)$ display jump discontinuities. Specifically, for any fixed parent cash-on-hand a^p there exists an upward jump in $A_{1,y}(\cdot, a^p)$ and $P_{0,s}(\cdot, a^p)$ at some level of child cash-on-hand a^k.*

Proof. Fix the parent's savings choice and write $a^p_{1,g} = a^p_{1,y} + y^p_1$. We can then replace y^p_1 by $a^p_{1,g}$ in the environment without parent savings and apply Prop. 2.1 to show the desired results. ■

In view of this result, it is unsurprising that in the initial gift-giving (plus parent-savings) stage similar dynamics as in the baseline model play out. Computationally, we find the same kinds of discontinuities in gifts as in the baseline setting. The dynamics are displayed in Fig. 4, which we generate using brute-force maximization on a fine discrete grid, thus ensuring that the algorithm can deal with value-function discontinuities. In the left upper corner of the graph, the parent is rich relative to the child and gives spoon-feeding gifts that are entirely consumed by the child. In the upper right corner, shots to autarky occur, taking the child exactly to the boundary of the autarky region. As in the baseline model, there is a stark discontinuity at the point where gifts-to-autarky commence: At the points where the diagonal arrows pertaining to shots-to-autarky emanate, gifts jump from zero to an amount proportional to the length of the diagonal arrow. Finally, in the bottom middle there is a region in which the dictator regime is played and no gifts occur in the first period.[22] Also for value functions, we find again the same smoothness properties as in the baseline model, i.e. various kinks and discontinuities.

4.2 Adding uncertainty

We now return to the baseline setting but include uncertainty over the child's endowment. In this subsection, assume that y_1^k is uncertain and realized before the parent makes the final-period gift decision. At this point we fully rely on numerical methods. Again, our point is that the key features of the deterministic setting remain intact. This is true even for unrealistically large levels of noise.

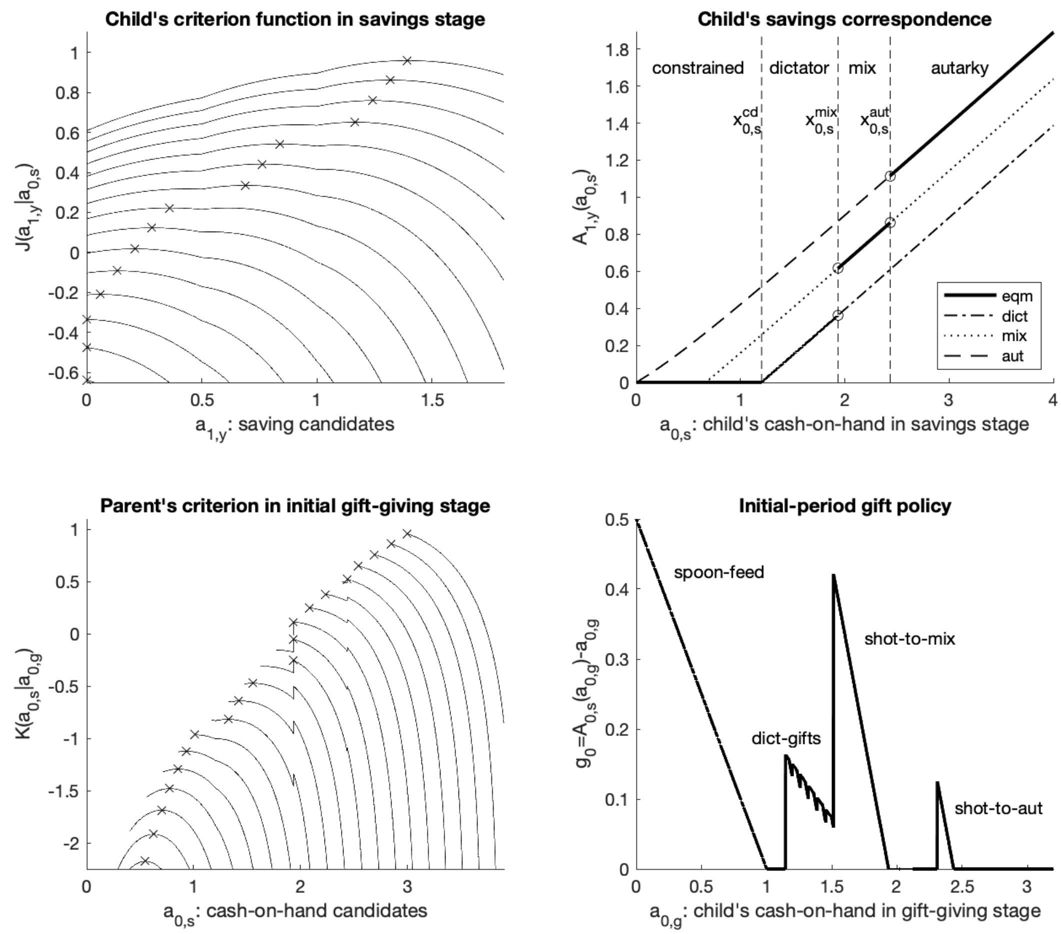
Discrete support. A common way of modeling income uncertainty in quantitative work is to assume that income shocks follow a discrete-state Markov process. Previous literature has conjectured that such noise is sufficient to smooth out non-convexities and the like. We now provide a simple example to show that a discrete-state process can make matters actually more complicated than in the deterministic setting, i.e. new types of regions can occur and the number of kinks and discontinuities can actually increase further. For simplicity, we will assume that there are only two income realizations, high and low, but the logic carries over to any number of finite states.

Fig. 5 shows the child's criterion and policy in the savings stage in the top panels in an example close to our baseline. As in the deterministic case, the child's payoff function is non-concave, leading to discontinuities in the child's savings policy and the parent's value function. In addition to the regions familiar from the baseline setting, there is now an additional region (*mixed*). In this region the child receives gifts only under the low (but not the high) income realization in the final stage. By the logic of the baseline model, the effective return to savings increases discretely as the child moves from the dictator- to the mixed- and then to the autarky-savings region, since the tax on gifts is removed in one state of the world each time. Hence, there are now two downward kinks in the child's criterion (shown in the lower-left panel), leading to *two* discontinuities in the child's savings policy. In the initial-period gift-giving stage, this translates into an additional regime (see lower right panel), thus increasing the number of jump discontinuities from two to three.

Continuous support In AHK, the income shock follows a continuous distribution, which gives us maximal hope that non-convexities are smoothed. But we now demonstrate that the jump discontinuities in policies and value functions (in stages without smoothing noise) remain intact, even when choosing very large levels of noise.

[22] Note that in the dictator regime, the parent's policy can in principle be indeterminate: The parent is indifferent between all combinations of gifts and savings that keep $a^p_{1,y} + g_0$ constant inside this region, since equilibrium policies in the remainder of the game only depend on the family's joint wealth, $a^p + a^k$. However, computationally we find no such indeterminacy in this example since all gifts into the dictator region involve zero savings for the parent.

Figure 5: Selected outcomes at $t = 0$ for two-state-support shock



Grid size: $N = 5,000$. Parameters: y^k_1 equals 0.5 with probability 1/2 and 0 otherwise; $u(c) = \ln c, \alpha = \beta = R = 1, y^p_0 = y^p_1 = 1$.

Suppose now that the child's income in the income stage of the final period follows a log-normal distribution, $\ln y_1^k \sim \mathcal{N}(\mu, \sigma^2)$. Figure 6 shows the situation when the expected value of child's income equals its baseline value (one-fourth) and its standard deviation equals one,[23] an empirically unrealistically high level of income uncertainty. We see that the implications of the deterministic model carry over: The child's payoff functions in the savings stage remain non-concave (and often double-peaked), which is due to the strong convexity in the parent's final-period gift-giving policy. This induces the same discontinuity in the child's savings policy and the parent's value function in this stage. Just as in the baseline model, this then feeds back and generates two jump discontinuities in the initial-period gift-giving policy.[24]

5 Implications of our theoretical findings

We now ask the following question: How far off would we be if we didn't solve the model correctly using the *global method* outlined previously? Specifically, how would the results differ if we solved the model based on first-order conditions, i.e. using a *local method*, which is the approach followed by much of the literature? This section provides a coarse lifecycle calibration of our baseline model to give a (tentative) answer to this question. Specifically, the *local method* can be summarized as follows:

1. In the gift-giving stage of the initial period, only transfers are considered that are fully consumed (i.e. spoon-feeding transfers). Thus, dictator gifts and shots to autarky are swept under the rug.

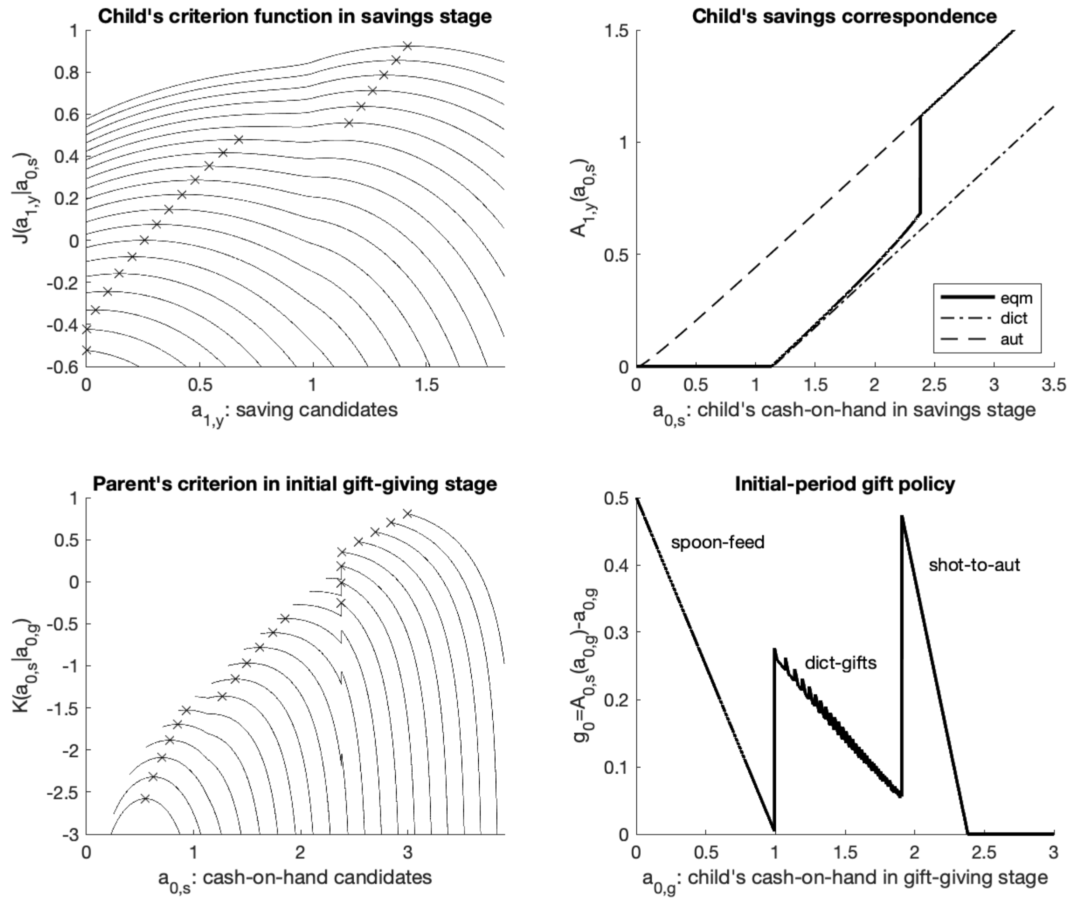
2. In the savings stage of the initial period, the child's savings choice is determined from the first-order condition: If the marginal benefit of saving is negative already for the first unit saved, then zero savings are chosen. Otherwise, the first zero of the first-order condition is selected.[25]

[23] I.e. we set $\sigma = 1.68$ such that $std(y_1^k) = 1$.

[24] We found that increasing the standard deviation of the child's endowments even further is insufficient to smooth out the payoff functions; we do not present these results as they are nearly indistinguishable from the case presented here. As Fig. C.2 in the appendix shows, however, *increasing the child's expected endowment* can render the child's criterion function concave and thus yield a continuous savings function. The reason is as in the deterministic case when Cond. 1 is violated: Final-period transfers become very unlikely. Interestingly, Fig. C.2 shows that even with a continuous savings function the parent's first-period criterion can be convex, since the child's savings function is, and the parent's gift policy can have a jump discontinuity.

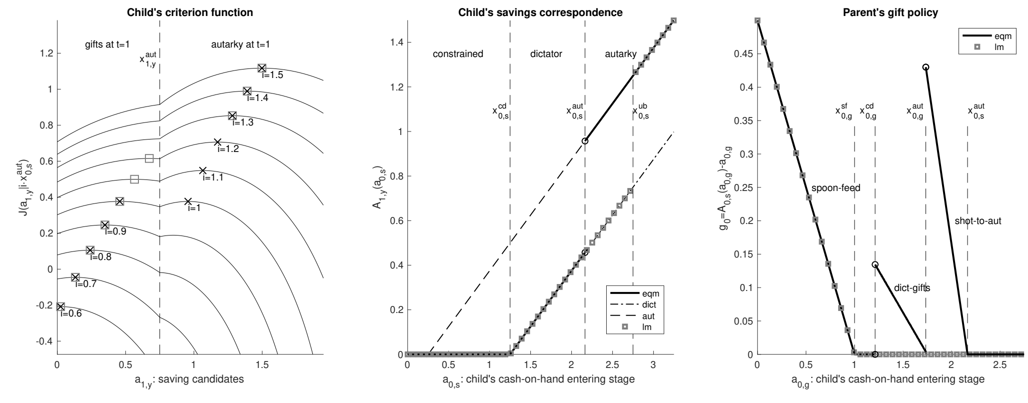
[25] We deem it most plausible that when searching for a root of the first-order condition, the algorithm starts the search at zero savings. Consequently, if there are multiple local maxima, we assume that the one with lowest savings is selected. In other words, we presume that the most common type of mistake in the savings stage is that i) the local minimum to the left of the kink in Fig. 7 is selected although it is not a global maximum. However, we note that algorithms based on Euler Equations may also commit the following types of errors: ii) pick up a local maximum to the right of the kink although it is not a global maximum or iii) pick up the local minimum at the kink, which may be smoothed by shocks or interpolation methods.

Figure 6: Selected outcomes at $t = 0$ for continuous-support shock



Grid size $N = 5,000$. Parameters: y_1^k drawn from log-normal distribution with mean 1/4 and standard deviation 1; $u(c) = \ln(c)$, $\alpha = \beta = R = 1$, $y_0^p = y_1^p = 1$.

Figure 7: Selected outcomes of local vs. global method at $t = 0$



Results from local method (*lm*) are indicated by squares and superimposed on results from global method (solid lines). Parameters as in baseline: $u(c) = \ln(c)$, $\alpha = \beta = R = 1$, $y_0^p = y_1^p = 1$, and $y_1^k = 1/4$.

3. In the gift-giving stage of the final period, both methodologies coincide: Transfers flow only if the standard spoon-feeding transfer motive is operative.

Figure 7 demonstrates how outcomes obtained by the local method differ from the correct global method, superimposing the results from the local-method algorithm as squares on previous graphs, which we obtained using the (correct) global method. The child's criterion function in the left panel shows that the local-method algorithm incorrectly selects dictator savings for longer, which implies more situations in which the child over-consumes and there are final-period parent transfers, i.e. the Samaritan's dilemma plays out. This means that the upward jump in the savings correspondence occurs later, as the middle panel shows. Finally, the right panel shows that by construction, under the local method first-period gifts flow only if the parent's static transfer motive is operative and the child is constrained, meaning that there are only transfers of the spoon-feeding type.

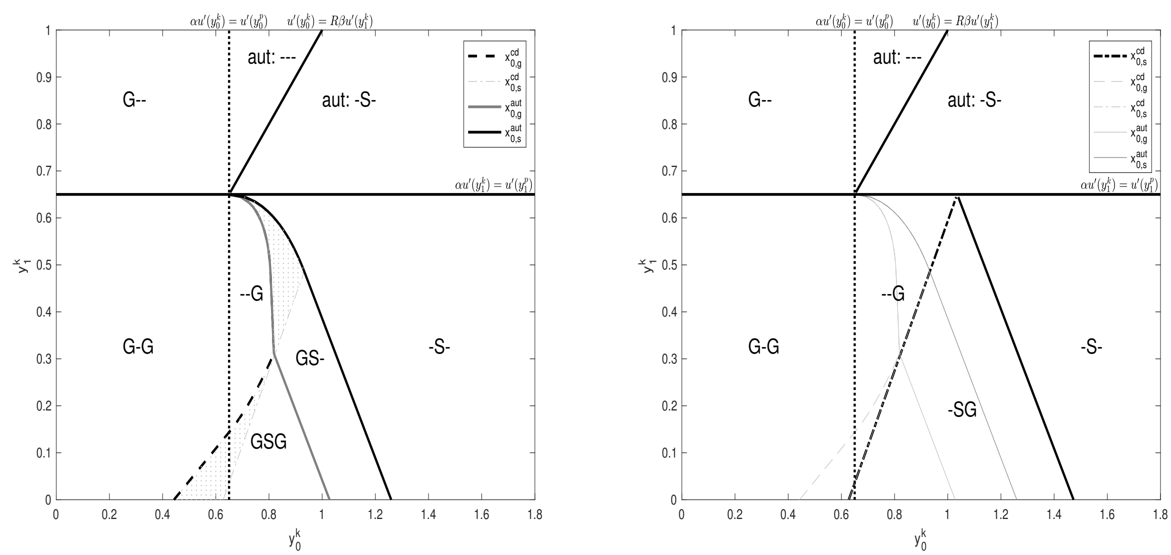
In the remainder of this section, we move to a more concrete example and adopt a lifecycle perspective. We consider a parent in their prime savings years and beyond (age 45 and above) and a child that is respectively younger (say, age 25 and above). In order to proxy that the parent should typically be able to smooth consumption in this age range and thus behave as under the permanent income hypothesis, we endow the parent with a constant earnings stream that we normalize to $y_0^p = y_1^p = 1$. Hence the child endowments y_0^k and y_1^k can be interpreted as multiples of the parent's permanent income in what follows.[26] We choose a model parameterization guided by standard lifecycle models. We use a power utility function with curvature $\gamma = 2$. Following Barczyk et al.

[26]We need not take a stand on the period length in Subsection 5.1; we will do so in Subsection 5.2.

(2021), we target an altruism coefficient of $\hat{\alpha} = 0.65$, which is defined as $\hat{\alpha} \equiv \alpha^{1/\gamma}$.[27] We set $R\beta = 1$, which is a standard assumption in deterministic life-cycle models.[28]

5.1 Comparing regimes under global and local method

Figure 8: Equilibrium regimes under global (left) and local (right) method



Note: $y_0^k = a_{0,g}$ = initial state. Left: Regions/regimes generated by correct global method separated by thick lines. Right: Regions generated by the erroneous local method separated by thick lines; thin gray lines demarcate regions under global method. Naming of regions gives type of equilibrium under equilibrium outcome given initial conditions; see text for naming convention. Shaded regions in left panel give loci where initial gifts act to alleviate borrowing constraints, i.e. points where the child would be constrained in the absence of an initial gift, but chooses positive savings in the equilibrium. Parameter values: power utility with $\gamma = 2$, $\hat{\alpha} = 0.65$, $R\beta = 1$, and $y_0^p = y_1^p = 1$.

Figure 8 shows the equilibrium regimes that the global (left) and local (right) method produce in the space of child's current- and future-period labor endowment for the given parameterization; the right panel also underlays the solution under the global method in light gray to facilitate the comparison.[29] We denote regions by a combination of letters and dashes, where the position of a letter or dash corresponds to the stage in the game, the order being given by: i. initial gift-giving, ii. savings, iii. final gift-giving; G means a positive gift flow, S means positive savings, and a dash (–) means that either G or S is zero. E.g. $--G$ means: i. no gift, ii. no savings, iii. gift.

We first note that the two methods coincide above the horizontal solid line, which is precisely the set of endowments for which **Cond. 1 fails**. These are cases excluded from our previous

[27] $\hat{\alpha}$ is the consumption ratio the parent would choose in a static setting if in charge of family resources.

[28] Note here that the results in Subsection 5.1, i.e. the shape of the regimes in Fig. 8, depends only on the product $R\beta$, but not on the values of R and β separately, as is evident from the relevant expressions in Appendix B.

[29] The shapes depicted in the figure are robust in the sense that they qualitatively remain unchanged when varying γ, α and $R\beta$ in a neighborhood.

analysis since they admit a straightforward solution as in a static model. Recall that Cond. 1 failing implies that even if the child chooses zero savings, there are no final-period gifts by the parent. To the left of the vertical dashed line the child obtains a current-period spoon-feeding transfer since $u'(y_0^p) < \alpha u'(y_0^k)$. Thus, in the upper-left quadrant (*G – – : spoon-feed to autarky*) the child obtains a gift in the initial but not in the final period; due to the back-loaded endowment profile the child does not save. The upper right quadrant contains the autarkic outcomes, i.e. configurations under which there are neither current nor future gifts. In this case the child can either be constrained (*aut: – – – : separate non-saver*) or unconstrained (*aut: – S – : separate saver*).

When **Cond. 1 holds** (the interesting case), however, we see that the regions generated by the two methods differ. Specifically, the right panel of Fig. 8 tells us that the local method misclassifies regimes most often in the left half of the lower right quadrant. These are situations in which i) the child has an intermediate to high current endowment but ii) a low-enough future endowment so that future transfers are at least a possibility. To understand the economics underlying the regions in the figure better and to link them to the previous figures, fix for now a relatively **small future child endowment** in the two panels of Fig. 8, e.g. $y_1^k = 0.1$. Now consider how the regimes change when increasing the value of the child's current endowment, y_0^k, starting from zero. Under both methods, there is initially spoon-feeding in both periods (*G–G*), the parent controlling the child's consumption and the child being constrained. The differences between the two methods then start to appear as we increase y_0^k: the child becomes unconstrained sooner under the global method ($y_0^k \simeq 0.6$) than under the local method ($y_0^k \simeq 0.7$). The reason is that under the correct global method, the parent gives a current-period transfer (*GSG: dictator savings*), whereas, under the local method this type of gift is absent, i.e. the *–SG* (*saving Samaritan*) area is entered. A key region then follows that is completely missed by the local method: shots to autarky (*GS–*, see Prop. 2.2), which is relatively large under our parameterization. The local method misconstrues these shots to autarky as *–SG* cases, meaning that the local method predicts lower savings and lower current transfers, but higher future transfers. Finally, we observe in the right panel that the region *–SG* under the local method erroneously contains part of the region where the child is actually in autarky (*–S–*, *save-to-autarky*), that is, the child saves and never receives gifts. The reason is that the global method detects earlier that autarky is optimal for the child than the local-method algorithm does.

The differences between local and global method play out somewhat differently, however, at **intermediate future child endowments**. Fix now a higher value, say $y_1^k = 0.5$, and vary again y_0^k starting at zero. The two methods now coincide exactly in the cases classified as permanent spoon-feeding (*G–G*). Under both methods, the region *– –G* (*constrained Samaritan*) then follows, in which the child receives no current gift, does not save, but then receives a future-period gift. This region is smaller under the global method, where shots to autarky (*GS –*) soon follow, this region being absent under the local method. The local method again misclassifies shots to autarky, this time either as constrained Samaritan or saving Samaritan (– – *G* or –*SG*). Finally, and similar to the case of low y_1^k, the child becomes autarkic (–*S*–) too late under the local method, the reason again being that the local method does not pick up the switch to the autarkic local optimum soon enough.[30]

To **summarize**, for both low and intermediate values of y_1^k (all satisfying Cond. 1), the local method generates too many constrained children, too few current transfers, but too many future transfers. In other words, the local method "over-emphasizes" the Samaritan's Dilemma, in which child savings are inefficiently low; incidentally (or not), this is in line with the focus of the early theoretical literature on altruism, which heavily stressed the Samaritan's Dilemma. Under the correct global solution, many Samaritan's-Dilemma situations are replaced by allocations in which the child is self-sufficient in the final period and child savings are actually efficient. The next subsection attempts to give an answer on how often we expect this to be the case.

5.2 A numerical exercise

For quantitative purposes, how far the local method is off will depend on how many data points fall into regions in Fig. 8 where the local solution differs from the global solution. In practice, this boils down to how many endowment combinations can be found in the left half of the lower-right quadrant. We now conduct a numerical exercise to obtain a tentative answer to this question. This experiment also gives us a rough quantitative sense on the implications for "how wrong" the local method gets the timing of transfers, gift amounts, and savings.[31]

We first note that our model has the drawback that there is only one period in which savings are possible, which makes the mapping to the data difficult. We deem it best to interpret the model as a coarse lifecycle framework in the spirit of traditional overlapping generations models with few life periods. We let one model period correspond to 20 years in the data. In $t = 0$, the child is of age 25 to 45 and the parent is of age 45 to 65; in $t = 1$ the child is of age 45 to 65 and the parent is of age 65 to 85.[32] We draw parent and child endowments from an income process that we construct so that

[30] The reader may wonder at this point why there is a discontinuity in the set of constrained agents at the thick horizontal line at which Cond. 1 starts to hold at $y_1^k \simeq 0.65$ in the right panel. The reason is as follows: If there are positive future transfers for some interval $[0, \epsilon)$, for ϵ ever so small, then the marginal benefit of savings evaluated at zero, i.e. at $a_1^y = 0$, is always multiplied by a constant that is strictly lower than one; intuitively, this constant embodies the "tax" on savings applied by an altruistic donor. This makes the marginal benefit of savings negative at zero (i.e. at $a_1^y = 0$) for an entire interval of values y_0^k (fixing y_1^k close to the Cond.-1 threshold). The local algorithm then erroneously concludes that zero savings are optimal, presuming that the marginal benefit of savings must be even lower for higher savings.

[31] Our calculations here are necessarily only suggestive due to the model's coarse time structure. Future work should study the model's implications in a quantitatively more credible framework, i.e. a model with a higher number of shorter time periods.

[32] Consequently, the model predictions have to be interpreted with caution: A zero transfer in period 0, for example,

Table 2: Case numbers of event sequences in numerical exercise

All draws *All regions*	Cond. 1	g_0-regime (see Cor. 2.1)	event sequencing: $g_0 > 0$	savings> 0	$g_1 > 0$	glob. mthd. cases (%)	loc. mthd. cases (%)
spoon-feed to autarky	×	aut	G	–	–	5.2	5.2
separate non-saver	×	aut	–	–	–	53.1	53.1
separate saver	×	aut	–	S	–	26.2	26.2
save to autarky	✓	aut	–	S	–	3.6	1.3
shot to autarky	✓	aut	G	S	–	1.6	∄
saving Samaritan	✓	wp	–	S	G	0.0	1.3
dictator savings	✓	wp	G	S	G	0.0	∄
constrained Samaritan	✓	cd	–	–	G	3.2	5.8
spoon-feeding	✓	cd	G	–	G	7.1	7.1

Event-sequencing codes are explained in the note to Figure 8. ∄ means a regime does not exist under local method, 0.0 means regime may exist but no cases occurred. Contains all draws ($N = 10^5$).

Table 3: Cross-tabulation of global against local method

Draws s.t. Cond. 1 (%) *Glob. mthd. regions*	*Local method regions* save to autarky	saving Samaritan	constrained Samaritan	spoon-feeding	sum
save to autarky	**8.4**	6.0	9.0	∄	23.4
shot to autarky	∄	2.2	8.0	∄	10.2
dictator savings	∄	0.3	0.1	0.0	0.4
constrained Samaritan	∄	∄	**20.3**	∄	20.3
spoon-feeding	∄	∄	∄	**45.7**	45.7
sum	8.4	8.5	37.4	45.7	100

Contains only draws for which Cond. 1 is satisfied. Entries in bold signify correct classifications by the local method.

Table 4: Child savings percentiles

Method	p50	p75	p90	p95	p99
global	0	0.06	0.26	0.37	0.61
local	0	0.03	0.22	0.35	0.61

Includes all draws.

Method	p50	p75	p90	p95	p99
global	0	0.20	0.35	0.42	0.59
local	0	0	0.11	0.38	0.59

Contains only draws for which Cond. 1 is satisfied.

Table 5: Fraction of all dollars transferred

	initial period					*final period*				
Method	GS-	GSG	G-G	G- -	sum	GSG	-SG	G-G	- -G	sum
global	6.3	0.2	33.5	15.2	55.2	0.5	0.0	32.6	11.7	44.8
local	∄	∄	32.8	15.0	47.8	∄	3.5	32.0	16.7	52.2

Percentage of all dollars transferred in initial period ($t = 0$) and final period ($t = 1$) discounted by $R = 1.02^{20}$. Contains all draws. For the first three columns Cond. 1 holds.

it is consistent with the (i) lifecycle profile of earnings, (ii) the persistence of earnings and (iii) the inter-generational correlation of earnings in U.S. data, see Appendix D for the details. We assume that both family members have perfect knowledge of all future earnings realizations.[33] In order to proxy that the parent is able to smooth consumption, we calculate parent permanent income for the given draw and then normalize all endowments by this number. We continue to use the parameters used for Fig. 8 above ($\gamma = 2$ and $\hat{\alpha} = 0.65$) and set a standard annual interest and discount rate of 2%, implying that $R = 1.02^{20} = 1/\beta$.

Table 2 shows which percentages among all draws fall within the various regimes in Fig. 8. Under our parameterization, Cond. 1 fails for 84.5% of all cases, in which case both methods coincide. However, these cases are mostly uninteresting in the sense that there is no parent-child interaction and gifts are zero except for the spoon-feed to autarky region, where initial transfers occur. When Cond. 1 holds, in turn, the local method assigns case numbers that are substantially different from the correct method for all regions except spoon-feeding (corresponding to the lower left quadrant in Fig. 8). Among all cases, the local method gets 4% wrong; however, this number increases to 19% when considering cases in which some transfer occurs and to 25% among Cond. 1 cases, suggesting important differences for the subset of families with parent-child interaction. The general pattern is that the local method understates autarkic outcomes (save-to-autarky and shots to autarky) and instead overstates outcomes in which the Samaritan's Dilemma plays out.

To see exactly where the local method goes wrong, Table 3 shows a cross-tabulation of how the two methods classify cases, focusing on draws for which Cond. 1 is satisfied. According to the global method, about one third of Cond.-1 children save in an autarkic fashion (entries *save to autarky* or *shot to autarky*), whereas such autarkic savings occur for only about 8% when following the local method. The local method misclassifies these autarkic cases as Samaritan's dilemmas (saving Samaritan and constrained Samaritan), a similar share of misclassifications stemming from shots to autarky as from save-to-autarky cases (under the correct solution).

How much do these case misclassifications matter for savings and transfer outcomes? Table 4 shows selected percentiles of child savings implied by the model. The lower part of the table tells us that solving the model correctly leads to large increases in savings among Cond. 1 cases, especially in the upper range of the savings distribution (but maybe not the very top). This then translates into more moderate –but still very substantial– increases of savings in the total population, as the upper part of the table shows. In relative terms, we observe a doubling of savings at percentile 75 and a

should be interpreted as there being "on average" no gifts from parent to child at child age 25 to 45. In reality, or in fact in a model with a finer time structure, however, there may be positive gifts from age 25 to 27 but then no gifts from 27 to 45. This word of caution adds to our plea for a more serious quantitative exploration in a model with more periods.

[33]Assuming uncertainty (at least) about the child's future earnings would certainly be more realistic, but is beyond the scope of the current paper and thus left for future research.

one-fifth increase at percentile 90. Also in absolute terms, the increases are substantial, amounting to an increase in mid-age savings on the order of one year of typical earnings.[34] Finally, Table 5 presents figures on the timing of transfers. It shows which percentage of all dollars transferred in both periods (discounted to $t = 0$) flows in $t = 0$ and $t = 1$, separating this number into contributions from the different regimes.[35] Solving the model correctly leads to more front-loading of transfers, increasing the fraction of initial transfers by almost one-fifth. This increase is almost entirely accounted for by shots to autarky (GS–), which the local method misses. The table shows again that the local method attributes too much importance to Samaritan-type transfers (–SG and – –G), almost doubling the percentage of such transfers under the correct solution.

The overall message from this numerical exercise is that solving the model correctly leads to substantial quantitative changes, especially in the subset of interesting cases with parent-child interaction. The Samaritan's Dilemma, which previous literature has focused on, loses importance in favor of autarkic allocations with efficient child savings. The quantitative errors are sizable already in a two-period model and deterministic endowments; we would expect errors to gain in importance when adding more periods and uncertainty, since the set of parent-child pairs that interact at some point of their lives grows.

5.3 Testable implications

What are testable implications that arise from the correctly solved model? At first, our results seem somewhat disheartening: the jagged transfer profile under the global solution (see Fig. 7) suggests that this theory is able to produce almost *any* pattern of transfers, which makes it a difficult theory to reject. However, at closer inspection it turns out that several robust features emerge that can in principle be tested.

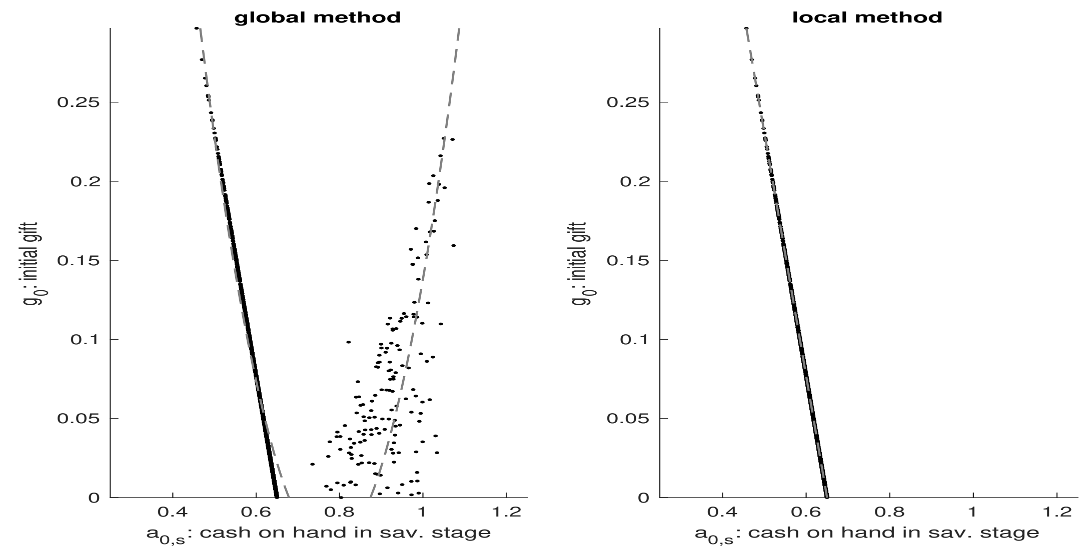
First, the global method predicts a U-shaped pattern when plotting gift size against cash-on-hand of the child, conditioning on cases with positive transfers.[36] We show this relation in Figure 9 for our numerical experiment and contrast it with the predictions of the local method. The correct solution implies initial-period transfers to wealth-poor as well as to wealth-rich children (relative to parents), the size of the transfer increasing on both ends when the child i) becomes substantially worse off than the parent or ii) when becoming as wealthy as the parent (i.e. close to 1). In the former case gifts are compensatory in nature, whereas in the latter case they serve as a kick-start to avoid moral hazard issues in the future.

[34]…bearing in mind that $0.05 = 1/20$ roughly corresponds to a typical yearly income, since one period stands in for 20 years and parent income is normalized to 1.

[35]The discounted sum of transfers is similar for both methods; we present percentages here to facilitate comparisons.

[36]We experimented with various degrees of utility curvature and altruism as well as with key parameters in the endowment process (i.e., the persistence of earnings, the intergenerational elasticity in earnings, and the variance of earnings) and reliably obtained the U-shaped pattern.

Figure 9: Positive gifts at $t = 0$



Positive initial gifts by parent, g_0, plotted versus child's cash-on-hand in savings stage, $a_{0,s}$ (i.e. including gifts). The total number of draws is $N = 10^4$; 1,363 obtain positive transfers under the global method and 1,210 under the local method. The dashed line in the global method is a second-order polynomial regression of gift size on child wealth with R-squared equal to 0.86. Model parameters as in numerical exercise in Section 5.2.

A second, closely related, implication is that the child's consumption growth and savings correlate positively with the front-loading of transfers. This is because children who receive shots to autarky display front-loaded transfers and save according to a standard Euler Equation. However, children who only receive transfers in the final period are subject to the Samaritan's dilemma and thus under-save. We note that this prediction is a general property of the model and thus independent of the parameterization in this section.

Finally, a well-known observable implication obtained from the local method that remains intact under the correctly-solved model is that altruistic transfers are such that they often flow to borrowing-constrained individuals. However, a caveat is that there are also first-period gifts to saving children under the correct solution.

6 Conclusions

In this paper, we have provided a full theoretical characterization of the basic two-period altruism model. Our results carry important consequences.

First, the way we think about and characterize operative transfer motives needs to be expanded beyond equalization of marginal utilities. Second, the statement that according to the altruistic hypothesis, richer children should receive smaller transfers, ceteris paribus, needs to be revised. Shots to autarky are a robust feature of equilibrium, meaning that it is entirely consistent with altruism that richer children can receive higher transfers. This prediction has been typically ascribed to the exchange-motivation hypothesis of transfers, the argument being that transfers to richer children must be larger in order to compensate them for services they provide. The argument also extends to how to empirically test the altruistic hypothesis. Third, the presence of uncertainty is unlikely to remove discontinuities in discrete-time altruism models; in fact, we have shown that uncertainty in the form of discrete shocks can make matters even worse. Fourth, the exclusion of parental savings does not fundamentally alter the nature of strategic interactions.

Furthermore, our results provide two inputs into a recent quantitative-macroeconomics literature that has used discrete-time altruism models with savings. First, our results can guide the quest for appropriate solution algorithms for these models (see our recommendations in Section 1). Second, our solutions for the power-utility case give a much-needed benchmark to test such algorithms and to judge their accuracy.

We leave it to future research to i) further identify the quantitative importance of the different types of transfers predicted by the model and ii) to find potential applications for the novel types of equilibria we identify, such as the college case in Section 3.1.

7 Acknowledgments

We thank the editor, Guillermo L. Ordoñez, and two anonymous referees for comments and suggestions that have substantially improved the paper. Daniel Barczyk acknowledges research funding through the SSHRC Insight Grant (435-2018-0754). Matthias Kredler acknowledges research funding from the Spanish Agencia Estatal de Investigación (grants RYC2018-025230-I and PID2019-107134RB-I00 / AEI / 10.13039/501100011033) and Comunidad de Madrid (Spain), grant EPUC3M11 (V PRICIT).

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A Proofs and additional results

Proof. (Lemma 2) Applying the Implicit Function Theorem to Eq. (5) and using it in Eq. (7), we find

$$A'_{1,s}(a_{1,g}) = \frac{u''(c_1^p)}{u''(c_1^p) + \alpha u''(c_1^k)} \in (0,1) \qquad \text{for } a_{1,g} < x_{1,g}^{aut},$$
$$\text{where } c_1^p = y_1^p + a_{1,g} - A_{1,s}(a_{1,g}) \qquad \text{and } c_1^k = A_{1,s}(a_{1,g}).$$

This derivative is continuous since $u''(\cdot)$ is a continuous function by Ass. 1. The slope being 1 for $a_{1,g} > x_{1,g}^{aut}$ follows from Eq. (6). The statement about the downward kink at $x_{1,g}^{aut}$, i.e. Eq. (10), also follows directly from $A_{1,s}(\cdot)$ being equal to $A_{1,s}^{dict}(\cdot)$ below $x_{1,g}^{aut}$ and equal to $A_{1,s}^{aut}(\cdot)$ above. The child's value coming into the stage is then given by $V_{1,g}(a_{1,g}) = u(A_{1,s}(a_{1,g}))$, which inherits the differentiability properties of $A_{1,s}(\cdot)$ by the Chain Rule and since $u(\cdot)$ is continuously differentiable by Ass. 1. As for the statement on the parent's value function, notice that $P_{1,g}^{dict}(\cdot)$ is an upper envelope to $P_{1,g}^{aut}(\cdot)$, since zero gifts are always a feasible option in the dictator problem. Hence,

$$P_{1,g}^{aut}(a_1) = u(y_1^p) + \alpha u(a_{1,g}) \leq P_{1,g}^{dict}(a_{1,g}) = u(y^p + a_{1,g} - A_{1,s}^{dict}(a_{1,g})) + \alpha u\big(A_{1,s}^{dict}(a_{1,g}))\big),$$

with equality only for $a_{1,g} = x_{1,g}^{aut}$. By the Envelope Theorem, we have $P_{1,g}^{aut\prime}(x_{1,g}^{aut}) = P_{1,g}^{dict\prime}(x_{1,g}^{aut})$. Thus the left and right derivative of $P_{1,g}(\cdot)$ at $x_{1,g}^{aut}$ coincide and $P_{1,g}(\cdot)$ is differentiable at $x_{1,g}^{aut}$. Since the child's consumption is increasing in $a_{1,g}$, $P'_{1,g}(a_{1,g})$ is monotone decreasing and thus $P_{1,g}(\cdot)$ is globally concave. Finally, concavity of $V_{1,g}(a_{1,g})$ on the range $(x_{1,g}^{aut}, \infty)$ follows from concavity of $u(\cdot)$ and the fact that the child consumes $a_{1,g}$ for $a_{1,g} > x_{1,g}^{aut}$. ■

Remark: Depending on the shape of the gift function, the child's value function may not be concave in the dictator region. Differentiating the child's value function $V_{1,g}(a) = u(A_{1,g}(a))$ twice and rearranging, we find that a sufficient condition for concavity on the dictator range is

$$\frac{A''_{1,s}(a)}{A'_{1,s}(a)} \leq -\frac{u''(A_{1,s}(a))}{u'(A_{1,s}(a))} \qquad \text{for all } a \in [0, a_{1,g}^{aut}], \tag{A.1}$$

i.e. the gift function "should not be too convex" and override the concavity of the utility function. Specifically, the condition says that the amount allotted to the child should grow at a rate below the growth rate of marginal utility at all points. For homothetic preferences (power utility), the condition is fulfilled since the marginal propensity to give is constant, i.e. $A''_{1,s} = 0$.

Proof. (Lemma 4) Since savings A are optimal given state a, saving ϵ less must do weakly worse, i.e. we have

$$0 \geq J(A - \epsilon; a) - J(A; a) \qquad \text{for all } \epsilon \in (0, A].$$

Now, writing out the terms of $J(\cdot)$ and using the fact that the marginal cost of savings drops as one becomes richer, we have for any $\epsilon \in (0, A]$:

$$
\begin{aligned}
0 \geq & u\big(a - A/R + \epsilon/R\big) - u(a - A/R) + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
= & \int_{a-A/R}^{a-A/R+\epsilon/R} u'(c)dc + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
> & \int_{a-A/R}^{a-A/R+\epsilon/R} u'(c+\delta)dc + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
= & \int_{a+\delta-A/R}^{a+\delta-A/R+\epsilon/R} u'(c)dc + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
= & u\big(a + \delta - A/R + \epsilon)/R\big) - u(a + \delta - A/R) + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
= & J(A - \epsilon; a + \delta) - J(A; a + \delta).
\end{aligned}
$$

for any $\delta > 0$. Note here that the strict inequality is justified since $u'(\cdot)$ is a strictly decreasing function by Ass. 1 and since $\delta > 0$. From the above it follows that $J(A - \epsilon; a + \delta) < J(A; a + \delta)$, i.e. $A - \epsilon$ is not optimal for state $a + \delta$. Since $\epsilon \in (0, A]$ and $\delta > 0$ were arbitrary, this establishes the desired result. ■

Proof. (Proposition 2.1) We first prove that as $a_{0,s} \to \infty$, the child will want to save enough to enter the autarky region in order to smooth consumption. Note that the derivative of the child's continuation value is

$$V'_{1,y}(a) = V'_{1,g}(a + y_1^k) = u'(c_1^k) A'_{1,s}(a + y_1^k).$$

The idea now will be to show that the marginal benefit of savings is fixed, while the marginal cost of savings approaches zero as the child gets richer. On the range where gifts are positive, we can bound $u'(c_1^k) \in [u'(x_{1,g}^{aut}), u'(A_{1,s}(y_1^k))]$ since the parent's gift policy $A_{1,s}(\cdot)$ is increasing by Lemma 2 and $u'(\cdot)$ is decreasing by Ass. 1. Again by Lemma 2, $A'_{1,s}(\cdot)$ is a continuous function on the interval $[0, x_{1,g}^{aut}]$ which satisfies $A'_{1,s}(a) \in (0, 1)$ for all $a \in [0, x_{1,g}^{aut}]$, thus there must exist bounds $0 < \min A'_{1,s} \leq \max A'_{1,s} < 1$ on this derivative by the Weierstrass Theorem. Thus the marginal continuation value $\beta V'_{1,y}(\cdot)$ is lower-bounded by $\beta u'(x_{1,g}^{aut}) \min A'_{1,s}$ in the dictator region, i.e. for $a \in [0, x_{1,y}^{aut}]$. However, as we let $a_{0,s} \to \infty$, the marginal cost of savings $u'(a_{0,s} - a'/R)$ approaches zero by the Inada condition in Ass. 1 for any fixed a' that leads into the transfer region. Hence we will have $J'(a'; a_{0,s}) > 0$ for all $a' \in [0, x_{1,y}^{aut}]$ for $a_{0,s}$ large enough, i.e. the child's criterion $J(\cdot; a_{0,s})$ will be increasing throughout the dictator regime . The optimal policy must thus feature autarky at $t = 1$ for $a_{0,s}$ large enough.

Now, denote the optimal savings policy in the autarky range by

$$\tilde{A}_{1,y}^{aut}(a_{0,s}) = \arg\max_{a' \geq x_{1,y}^{aut}} J(a'; a_{0,s})$$

for all $a_{0,s} \geq x_{1,y}^{aut}/R$, i.e. for all states for which saving into autarky is feasible. Since J is continuous and strictly concave on the autarky range by Lemma 3, the maximum is attained by a unique maximizer for each state $a_{0,s}$, so $\tilde{A}_{1,y}^{aut}(\cdot)$ is a singleton-valued correspondence and thus a function. By Berge's Maximum Theorem, $\tilde{A}_{1,y}^{aut}(\cdot)$ is also continuous. Now, denote the lowest state at which autarky is among the child's optimal policies by

$$x_{0,s}^{aut} \equiv \inf \left\{ a_{0,s} : \tilde{A}_{1,y}^{aut}(a_{0,s}) \in A_{1,y}(a_{y,s}) \right\}.$$

It then follows from increasingness of the savings policy (Lemma 4) that any optimal policy must feature autarky for any state $a_{0,s} > x_{0,s}^{aut}$. As shown above, the savings policy $A_{1,y}(\cdot)$ must thus be a continuous function on the range $a_{0,s} > x_{0,s}^{aut}$. Again by Berge's Maximum Theorem, $\tilde{A}_{1,y}^{aut}(x_{0,s}^{aut})$ must also be optimal at the threshold $x_{0,s}^{aut}$, since upper-hemi-continuity of the optimal policy translates into continuity for a function (or singleton-valued correspondence). Furthermore, upper-hemi-continuity and non-emptiness of the policy correspondence (which are again guaranteed by the Maximum Theorem) on the range $a_{0,s} < x_{0,s}^{aut}$ imply that there must also be a second maximizer $a' \in A_{1,y}(x_{0,s}^{aut})$ with $a' < x_{1,y}^{aut}$ that leads into the dictator regime at the threshold. The value function $V_{0,s}(\cdot)$ is continuous at the threshold $x_{0,s}^{aut}$ and the child is indifferent between the (best) autarkic and the (best) dictator-savings policy, again by the Maximum Theorem.

Also, note that for low enough child cash-on-hand, autarky is not an option since at some point it is not feasible to save into this area. Formally, when $a_{0,s} \to 0$, autarky is not feasible if Condition 1 holds. We also must have that zero savings are optimal for $a_{0,s}$ low enough.

By the Maximum Theorem, the child's value function $V_{0,s}(\cdot)$ is continuous at $x_{0,s}^{aut}$; we will now show that the parent's value function is discontinuous at this point, however. Denote by a^{dict} the maximal amount that the child saves in the dictator region, i.e. set $a^{dict} = \max\{A_{1,y}(a) : a \in [0, x_{0,s}^{aut}]\}$; note that the maximum here is attained by the Maximum Theorem. Lemma 4 tells us that a^{dict} must be an optimal savings policy at the autarky threshold, i.e. $a^{dict} \in A_{1,y}(x_{0,s}^{aut})$. Also, it must be that a^{dict} takes the economy within the dictator region and not on the kink, i.e. $a^{dict} < x_{1,y}^{aut}$, since the criterion $J(\cdot)$ has a downward kink at the threshold. But this implies that the parent will give a positive gift and consumption will be strictly lower than under autarky. Now, since the parent's value function equals $P_{0,s}(a) = \alpha V_{0,s}(a) + \beta u(c_1^p)$, where $V_{0,s}(\cdot)$ is continuous and c_1^p jumps, the parent's value function has an upward jump when the regime switches to autarky, i.e.

we have

$$\sup_{a<x_{0,s}^{aut}} P_{0,s}(a) < \inf_{a>x_{0,s}^{aut}} P_{0,s}(a), \tag{A.2}$$

which concludes the proof. ■

Remark: Note that the proof does not make a statement about which policy is chosen at the threshold $x_{0,s}^{aut}$ itself. Since both dictator and autarkic savings are optimal for the child, either of the two regimes can be played in equilibrium. Also, note that the parent's value function (but not the child's value function) may have further discontinuities within the dictator region since the child may switch from one local maximum to another. Such jumps in savings must always be upward, all resulting jumps in the parent's value must also be upward by the same argument as above.

Proof. (Lemma 5) This proof follows exactly the same strategy as the proof of Lemma 4, but taking care of the non-negativity constraints for gifts.

Fix some child cash-on-hand $a \geq 0$ coming into the gift-giving stage at $t = 0$. First, note that if giving zero gifts is optimal, i.e. $A = a \in A_{0,s}(a)$, then the statement follows trivially since a is not feasible for any state $a + \delta$, for $\delta > 0$, since gifts cannot be negative.

So assume from now on that $A > a$, i.e. the gift is positive. Again, note that the statement in the lemma follows trivially for any δ large enough such that A is not feasible any more, i.e. $a+\delta > A$.

Thus restrict attention to δ small enough such that $a + \delta \leq A$, i.e. setting A is feasible at state $a + \delta$. Since A is optimal at a for the parent, setting ϵ less must do weakly worse, i.e. we have

$$0 \geq K(A - \epsilon; a) - K(A; a) \qquad \text{for all } \epsilon \in (0, A - a],$$

where we recall that $K(A; a)$ denotes the parent's payoff of setting the child's cash-on-hand to $a_{0,s} = A$ given state $a_{0,g} = a$. Now, writing out the terms of K and using the fact that the marginal cost of savings drops as one becomes richer, we have for any $\epsilon \in (0, A - a]$:

$$\begin{aligned} 0 \geq & u\big(a - A + \epsilon\big) - u(a - A) + P_{0,s}(A - \epsilon) - P_{0,s}(A) \\ > & u(a + \delta - A + \epsilon) - u(a + \delta - A) + P_{0,s}(A - \epsilon) - P_{0,s}(A) \\ = & K(A - \epsilon; a + \delta) - K(A; a + \delta) \end{aligned}$$

for any $\delta > 0$. As in the proof for Lemma 4, the strict inequality is justified since $u(\cdot)$ is strictly concave by Ass. 1 and since $\delta > 0$. From the above it follows that $K(A-\epsilon; a+\delta) < K(A; a+\delta)$, i.e. $A - \epsilon$ is not optimal for state $a + \delta$, which completes the proof. ■

Proposition A.1 (Parent's marginal benefit from initial-period gifts)**.** *Whenever $A'_{1,y}(a_{0,s})$ exists*[37]*, the parent's marginal benefit of gift-giving in the initial period is given by*

$$P'_{0,s}(a_{0,s}) = \alpha u'(c_0^k) + \mathbb{I}\{a_{0,s} \in (x_{0,s}^{cd}, x_{0,s}^{aut})\}\beta u'(c_1^p) \underbrace{\left[1 - A'_{1,s}(a_{1,g})\right]}_{=c_1^{p\prime}(a_g^1)\in(0,1)} \underbrace{A'_{1,y}(a_{0,s})}_{>0}, \tag{A.3}$$

where $c_0^k = a_{0,s} - A_{1,y}(a_{0,s})/R$, $a_{1,g} = A_{1,y}(a_{0,s}) + y_1^k$ and, following up, $c_1^p = y_1^p + a_{1,g} - A_{1,s}(a_{1,g})$ and $c_1^k = A_{1,s}(a_{1,g})$ are understood to be the outcomes on the equilibrium path of the game.

Proof. (Prop. A.1) Whenever $A'_{1,y}(a_{0,s})$ exists, we can write

$$\begin{aligned} P'_{0,s}(a_{0,s}) &= \alpha u'(c_0^k)\left[1 - A'_{1,y}(a_{0,s})/R\right] + \beta P'_{1,y}\big(A_{1,y}(a_{0,s})\big)A'_{1,y}(a_{0,s}) \\ &= \alpha u'(c_0^k)\left[1 - A'_{1,y}(a_{0,s})/R\right] + \alpha\beta u'(c_1^k)A'_{1,y}(a_{0,s}) \\ &= \alpha u'(c_0^k) - \alpha A'_{1,y}(a_{0,s})\left[u'(c_0^k)/R - \beta u'(c_1^k)\right]. \end{aligned} \tag{A.4}$$

When going from the first to the second line, we substitute $P'_{1,y}(A_{1,y}) = P'_{1,g}(A_{1,y} + y_1^k) = \alpha u'(c_1^k)$, which follows from Eq. (11) in Lemma 2; the third line only groups terms. We will now show that we can further simplify this expression to obtain (A.3).

First, notice that when the child is constrained, then there is no savings response and we have $A'_{1,y}(a_{0,s}) = 0$ in (A.4); the parent's marginal value from the gift is then fully captured by the child's current marginal utility and $P'_{0,s}(a_{0,s}) = \alpha u'(c_0^k)$, as claimed. Similarly, if the child chooses autarkic savings, then the Euler Equation $u'(c_0^k) = R\beta u'(c_1^k)$ must hold, and again the parent's marginal value is $P'_{0,s}(a_{0,s}) = \alpha u'(c_0^k)$, as claimed. However, when the child engages in dictator-savings, we need to take care of the distortion that future gifts introduce in the child's optimality condition. Using first the child's distorted Euler equation (14) and then the parent's first-order condition for final-period gifts (5), one obtains

$$\begin{aligned} P'_{0,s}(a_{0,s}) &= \alpha\left(u'(c_0^k) - [\beta u'(c_1^k)A'_{1,s}(a_{1,g}) - \beta u'(c_1^k)]A'_{1,y}(a_{0,s})\right) \\ &= \alpha\left(u'(c_0^k) + \beta u'(c_1^k)[1 - A'_{1,s}(a_{1,g})]A'_{1,y}(a_{0,s})\right) \\ &= \alpha u'(c_0^k) + \beta u'(c_1^p)[1 - A'_{1,s}(a_{1,g})]A'_{1,y}(a_{0,s}), \end{aligned}$$

which is the correction term in (A.3) that is only active in the dictator-savings region. ■

Remark: In the dictator region, the first component of the parent's marginal value in (A.3) is

[37] The derivative $A'_{1,y}$ surely exists inside the constrained and autarkic regimes, since $A'_{1,y}(a_{0,s}) = 0$ for $a_{0,s} < x_{0,s}^{cd}$ on the constrained range and since for $a_{0,s} > x_{0,s}^{aut}$ on the autarkic range the derivative $A'_{1,y}(a_{0,s}) = A_{1,y}^{aut\prime}(a_{0,s})$ exists by standard arguments. In the dictator-savings regime, i.e. for $a_{0,s} \in (x_{0,s}^{cd}, x_{0,s}^{aut})$, differentiability is not assured in general, but exists for power utility. Finally, at the thresholds $a_{0,s} \in \{x_{0,s}^{cd}, x_{0,s}^{aut}\}$ the derivative does not exist.

again given by the marginal utility of child's current-period consumption; however, the correction term tells us that we also have to take into account how much more the parent can consume tomorrow (captured by $c_1^{p\prime}$) due to the fact that the child saves more today (captured by $A'_{1,y}(a_{0,s})$). The correction term is always positive, since it is an additional benefit the parent derives from gift-giving. The correction term helps us to understand the downward kink in the parent's criterion in the upper-left panel of Fig. 3 at point $x_{0,s}^{cd}$, where the child switches from being constrained to dictator savings: The parent suddenly has an additional incentive to give and the slope of K changes when the regime switches.

Proposition A.2 (Interiority of gifts-to-autarky)**.** *Suppose Ass. 1 and Cond. 1 hold and denote by $c_0^{k,aut}(\cdot)$ the child's consumption policy in the savings stage under the autarky regime. Consider gifts leading into autarky in the initial-period gift-giving stage.*

1. *(**point landings**) If $\alpha u'(c_0^{k,aut}(x_{0,s}^{aut})) \leq u'(y_0^p)$, then all such gifts shoot the child exactly to the boundary of the autarky region. In other words, there is always a boundary solution to the parent's problem: $A_{0,s}(a) > a$ and $A_{0,s}(a) \geq x_{0,s}^{aut}$ imply $A_{0,s}(a) = x_{0,s}^{aut}$.*
2. *(**interior solutions**) If $\alpha u'(c_0^{k,aut}(x_{0,s}^{aut})) > u'(y_0^p)$, then (i) some shots to autarky go into the interior of the autarky region and (ii) there exist positive gifts for some starting conditions above the child's autarky threshold. To be precise, there exists an interval $\mathcal{I} = (x_{0,s}^{aut} - \epsilon_1, x_{0,s}^{aut} + \epsilon_2)$, for some $\epsilon_1 > 0$ and $\epsilon_2 > 0$, such that $A_{0,s}(a) > x_{0,s}^{aut}$ and $A_{0,s}(a) > a$ for all $a \in \mathcal{I}$, i.e. gifts are interior solutions.*

*(**Sufficient condition for point landings**) Furthermore, if $u'(y_0^p) \geq \beta R u'(y_1^p)$ – i.e. if the parent would not want to save at the market rate R – then only Case 1 is possible .*

Proof. (Proposition A.2) The parent's payoff from setting $A \geq x_{0,s}^{aut}$ given state a in the autarky region is $K(A;a) = u(y_0^p + a - A) + \alpha V_{0,s}^{aut}(A) + \beta u(y_1^p)$. Applying the Envelope Theorem to the child's value in autarky, the derivative of this function is given by

$$K_A(A;a) = -u'(\underbrace{y_0^p + a - A}_{=c_0^p}) + \alpha u'(\underbrace{A - A_{1,y}^{aut}(A)/R}_{=c_0^{k,aut}(A)}), \tag{A.5}$$

where the first term captures the marginal cost of giving (which is increasing in the gift A) and the second captures its marginal benefit (which is decreasing in the gift A since the child's autarkic problem is a standard savings problem in which consumption increases in initial assets).

Case 1: First, consider the marginal payoff of giving zero gifts, which we define as $K_A^0(a) \equiv K_A(a;a)$. The function $K_A^0(\cdot)$ is decreasing for $a \geq x_{0,s}^{aut}$ since $c_0^{k,aut}(\cdot)$ is a strictly increasing function. Now, the condition for Case 1 in the proposition implies $K_A^0(x_{0,s}^{aut}) \leq 0$, which then means that $K_A^0(a) < 0$ for any $a > x_{0,s}^{aut}$, i.e. the marginal benefit of giving the first gift dollar is already negative. Since $K_A(A;a)$ is decreasing in A, the marginal benefit of giving must then be negative for all feasible $A > x_{0,s}^{aut}$, which directly implies that any gift to autarky must be a corner solution as described in the proposition.

Case 2: Conversely, if the condition for Case 2 in the proposition holds, then for pairs (A, a) close to $(x_{0,s}^{aut}, x_{0,s}^{aut})$, we have $K_A(A, a) > 0$ by continuity of the functions $u'(\cdot)$ and $c_0^{k,aut}(\cdot)$ – recall again that the autarkic problem is a standard savings problem. Hence, for $\epsilon_1 > 0$ small enough a shot to autarky must occur by Prop. 2.2 and we have $K_A(x_{0,s}^{aut}, a) > 0$ for all $a \in (x_{0,s}^{aut} - \epsilon_1, x_{0,s}^{aut})$. Thus the shot must go into the autarky region. Second, for $\epsilon_2 > 0$ small enough we have $K_A(a;a) > 0$ for all $a \in [x_{0,s}^{aut}, x_{0,s}^{aut} + \epsilon_2)$, which implies that a positive gift is given. This concludes the proof of the claims in Case 2.

Sufficient condition for point landings. To show the last claim in the proposition, observe that

$$\alpha u'\big(x_{0,s}^{aut} - A_{1,y}(x_{0,s}^{aut})/R\big) = \alpha\beta R u'\big(A_{1,y}(x_{0,s}^{aut}) + y_1^k\big) \leq \beta R u'(y_1^p), \tag{A.6}$$

where the equality uses the child's Euler Equation for autarkic savings and the inequality uses the parent's first-order condition for gifts in the final period (which must be zero in autarky). If $u'(y_0^p) \geq \beta R u'(y_1^p)$, then (A.6) implies $u'(y_0^p) \geq \alpha u'(x_{0,s}^{aut} - A_{1,y}(x_{0,s}^{aut})/R)$ and thus $K_A(x_{0,s}^{aut}; x_{0,s}^{aut}) \leq 0$, i.e. the parent's marginal payoff from giving to the child when starting at the autarky threshold is negative (or zero), which is precisely the condition needed to guarantee that Case 1 occurs. ∎

B Solution for power utility

This appendix shows the solution for the power-utility case, i.e. invoking Assumption 2.

B.1 Auxiliary problems

We will first define the auxiliary problems. For the case of power utility, all of the auxiliary problems are differentiable and concave and the resulting unconstrained policies are affine functions, which is a result of homotheticity of preferences.

Dictator setting. Consider first the setting in which the parent dictates the consumption allocation in the final period. In the savings stage, recall here that we assume that the child can borrow against future family wealth to get rid of corner solutions.

Final period. In the final period, we can obtain the parent's dictator policy from (5) to obtain the policy

$$A_{1,s}^{dict}(a_{1,g}) = MPG^{sf}(y_1^p + a_{1,g}), \tag{B.1}$$

$$\text{where} \quad MPG^{sf} = \alpha^{1/\gamma}/(1+\alpha^{1/\gamma}), \quad a_{1,g} = y_1^k + a_{1,y}, \tag{B.2}$$

i.e. the parent allocates a fixed fraction MPG^{sf} of joint resources, $y_1^p + a_{1,g}$, to the child. MPG^{sf} is the *marginal propensity to give* when *spoon-feeding*, which will show up in the initial period as well.

Savings stage. Given the parent's decision rule in the final period, the child's problem in the savings stage is given by

$$\begin{aligned}
&\max_{c_0^{k,dict}, c_1^{k,dict}, a_{1,y}} \left\{ \frac{(c_0^{k,dict})^{1-\gamma} - 1}{1-\gamma} + \beta \frac{(c_1^{k,dict})^{1-\gamma} - 1}{1-\gamma} \right\} \\
\text{s.t} \quad & c_0^{k,dict} + \frac{a_{1,y}}{R} = a_{0,s}, \\
& c_1^{k,dict} = MPG^{sf}(y_1^p + y_1^k + a_{1,y}).
\end{aligned}$$

A key insight is now that this problem can be converted to a standard savings problem with a modified interest rate and modified endowments. Denote $\tilde{y}_1^k = MPG^{sf}(y_1^p + y_1^k)$, $\tilde{a}_{1,y} = MPG^{sf} a_{1,y}$ and $\tilde{R} = MPG^{sf} R$ and write the above problem equivalently as

$$\begin{aligned}
&\max_{c_0^{k,dict}, c_1^{k,dict}, \tilde{a}_{1,y}} \left\{ \frac{(c_0^{k,dict})^{1-\gamma} - 1}{1-\gamma} + \beta \frac{(c_1^{k,dict})^{1-\gamma} - 1}{1-\gamma} \right\} \\
\text{s.t} \quad & c_0^{k,dict} + \frac{\tilde{a}_{1,y}}{\tilde{R}} = a_{0,s}, \\
& c_1^{k,dict} = \tilde{y}_1^k + \tilde{a}_{1,y}.
\end{aligned}$$

The solution is standard and given by

$$\begin{aligned}
c_0^{k,dict} &= \underbrace{(1 + \beta^{1/\gamma} \tilde{R}^{(1-\gamma)/\gamma})^{-1}}_{\equiv MPC^{dict}} \tilde{W}_0, \\
c_1^{k,dict} &= (1 - MPC^{dict}) \tilde{R} \tilde{W}_0, \\
A_{1,y}^{dict}(a_{0,s}) &= -MPC^{dict} \cdot (y_1^p + y_1^k) + (1 - MPC^{dict}) \cdot R a_{0,s},
\end{aligned} \tag{B.3}$$

where we define $\tilde{W}_0 = a_{0,s} + \frac{\tilde{y}_1^k}{\tilde{R}}$ to be the present value of the (modified) endowments. MPC^{dict} is the child's *marginal propensity to consume* in the *dictator* regime. The child's value function in the savings stage of the dictator setting is given by

$$V_{0,s}^{dict}(a_{0,s}) = \frac{(c_0^{k,dict})^{1-\gamma} - 1}{1-\gamma} + \beta \frac{(c_1^{k,dict})^{1-\gamma} - 1}{1-\gamma}. \tag{B.4}$$

Initial gifts. The parent's gift-giving problem at $t = 0$, knowing that the dictator game will ensue, is

$$P_{0,g}^{dict}(a) \equiv \max_{A \in [0, a+y_0^p]} \left\{ u(a + y_0^p - A) + P_{0,s}^{dict}(A) \right\}, \tag{B.5}$$

where $P_{0,s}^{dict}(a)$ is the parent's value entering the child's savings stage described before. Denote the policy correspondence that solves this problem by $A_{0,s}^{dict}(a)$. Since the child's savings function is affine, this is a concave problem that we can solve in closed form as

$$A_{0,s}^{dict}(a) = \frac{B(a + y_0^p) - R^{-1}(y_1^p + y_1^k)}{1+B}, \tag{B.6}$$
$$\text{where} \quad B = \left[\alpha(MPC^{dict})^{1-\gamma} + \beta R^{1-\gamma}(1 - MPG^{sf})^{-\gamma}(1 - MPC^{dict})^{1-\gamma}\right]^{1/\gamma}.$$

Autarkic setting. Recall that in autarky, we force final-period gifts to be zero. Again, we let the child borrow against its future endowment to guarantee interior solutions.

Savings stage. For the child, we have again a standard two-period savings problem with solution

$$\begin{aligned} c_0^{k,aut} &= \underbrace{(1 + \beta^{1/\gamma} R^{(1-\gamma)/\gamma})^{-1}}_{=MPC^{aut}} W_0, \\ c_1^{k,aut} &= (1 - MPC^{aut}) R W_0, \\ A_{1,y}^{aut}(a_{0,s}) &= -MPC^{aut} \cdot y_1^k + (1 - MPC^{aut}) \cdot R a_{0,s}, \end{aligned} \tag{B.7}$$

where $W_0 = a_{0,s} + \frac{y_1^k}{R}$ is the present value of the endowment. The child's value function in the savings stage setting is

$$V_{0,s}^{aut}(a_{0,s}) = \frac{(c_0^{k,aut})^{1-\gamma} - 1}{1-\gamma} + \beta \frac{(c_1^{k,aut})^{1-\gamma} - 1}{1-\gamma}. \tag{B.8}$$

Initial gifts. The parent's problem at $t = 0$, knowing that the autarkic allocation will ensue, is given by

$$P_{0,g}^{aut}(a) \equiv \max_{A \in [0, a+y_0^p]} \left\{ u(a + y_0^p - A) + \alpha V_{0,s}^{aut}(A) \right\} + \beta u(y^p), \tag{B.9}$$

with the maximizer denoted by $A_{0,s}^{aut}(a)$. Note that this is a concave problem, since the function inside the curly brackets is concave. Algebra gives us the optimal policy

$$A_{0,s}^{aut}(a) = \frac{\alpha^{1/\gamma}}{MPC^{aut} + \alpha^{1/\gamma}}(a + y_0^p) - \frac{MPC^{aut}}{MPC^{aut} + \alpha^{1/\gamma}} \frac{y_1^k}{R}. \tag{B.10}$$

Constrained setting. Finally, consider a setting in which we force the child to consume all of its cash-on-hand in the savings stage at $t = 0$. Given child cash-on-hand a entering the game, the parent's *constrained problem* is given by

$$P_{0,g}^{cd}(a) \equiv \max_{A \in [0, a + y_0^p]} \left\{ u(a + y_0^p - A) + \alpha u(A) \right\} + \beta P_{1,y}(0), \tag{B.11}$$

where we allow negative transfers in order to eliminate corner solutions. Denote the optimal policy in this problem by $A_{0,s}^{cd}(a)$. Note that this is a static problem since the continuation value $P_{1,y}(0)$ is, by construction, always the one that ensues when the child has no savings and thus constant. Thus, the parent's optimal policy in the initial period is identical to the spoon-feeding policy that we already obtained for the final period:

$$A_{0,s}^{cd}(a) = MPG^{sf}(a + y_0^p), \tag{B.12}$$

where we recall that $MPG^{sf} = \alpha^{1/\gamma}/(1 + \alpha^{1/\gamma})$ is the marginal propensity to give. We will later also need the child's value function in the savings stage of the constrained setting, which under Cond. 1 is given by

$$V_{0,s}^{cd}(a) = \frac{(a^{1-\gamma} - 1}{1-\gamma} + \beta \frac{\left(MPG^{sf}(y_1^p + y_1^k)\right)^{1-\gamma} - 1}{1-\gamma}. \tag{B.13}$$

B.2 Solving the actual game

Final period. In the final-period's gift-giving stage, we find the threshold between the dictator and autarky regimes from Eq. (8) in closed form as

$$x_{1,g}^{aut} = \alpha^{1/\gamma} y_1^p. \tag{B.14}$$

Using Eq. (7) and (B.1), the equilibrium gift-giving policy is thus piecewise linear and given by

$$A_{1,s}(a_{1,g}) = \max\left\{ MPG^{sf}(y_1^p + a_{1,g}), a_{1,g} \right\}. \tag{B.15}$$

Savings stage. We first repeat here the expressions for the child's marginal propensity to consume within regions, since they will show up in the formulae below:

$$MPC^{dict} = \frac{1}{1+\beta^{1/\gamma}\tilde{R}^{(1-\gamma)/\gamma}}, \tag{B.16}$$

$$MPC^{aut} = \frac{1}{1+\beta^{1/\gamma}R^{(1-\gamma)/\gamma}}, \tag{B.17}$$

$$\text{where } \tilde{R} = \frac{\alpha^{1/\gamma}}{1+\alpha^{1/\gamma}}R. \tag{B.18}$$

We can now obtain the threshold $x_{0,s}^{cd}$ where the child's dictator savings become positive by solving $A_{1,y}(x_{0,s}^{cd}) = 0$. Using Eq. (B.3), this gives us the closed form

$$x_{0,s}^{cd} = \left[MPC^{dict}/(1-MPC^{dict})\right](y_1^p + y_1^k)/R. \tag{B.19}$$

We obtain the threshold at which autarky is entered, $x_{0,s}^{aut}$, by solving

$$\begin{aligned} &\mathbb{I}(a < x_{0,s}^{cd})V_{0,s}^{cd}(a) + \mathbb{I}(a \geq x_{0,s}^{cd})V_{0,s}^{dict}(a) = V_{0,s}^{aut}(a), \quad a \in [\underline{x}, \bar{x}], \\ &\text{where} \quad \underline{x} = \left(x_{1,y}^{aut} + MPC^{aut}y_1^k\right) / \left(R(1-MPC^{aut})\right), \\ &\qquad\qquad \bar{x} = \left(x_{1,y}^{aut} + MPC^{dict}(y_1^p + y_1^k)\right) / \left(R(1-MPC^{dict})\right), \end{aligned} \tag{B.20}$$

where $x_{0,s}^{cd}$ is given by Eq. (B.19), $V_{0,s}^{cd}(a)$ is given by Eq. (B.13), $V_{0,s}^{dict}(a)$ is given by Eq. (B.4), and $V_{0,s}^{aut}(a)$ is given by Eq. (B.8). The bounds $\{\underline{x}, \bar{x}\}$ are useful for computation and are derived as the maximal/minimal asset level under which the child's criterion is double-peaked, which can be found as the asset levels under which the child's criterion reaches a local maximum exactly at the autarky threshold $x_{1,y}^{aut}$ in each of the two regions.

The optimal savings policy is then given by

$$A_{1,y}(a) = \max\left\{0, \mathbb{I}(a < x_{0,s}^{aut})A_{1,y}^{dict}(a) + \mathbb{I}(a \geq x_{0,s}^{aut})A_{1,y}^{aut}(a)\right\}$$

where $x_{0,s}^{aut}$ satisfies Eq. (B.20), $A_{1,y}^{dict}(a)$ is given by Eq. (B.3) and $A_{1,y}^{aut}(a)$ by Eq. (B.7). This leads to the following proposition:

Proposition B.1 (Child's marginal propensity to consume)**.** *Under Ass. 2 (power utility), the marginal propensities to consume in the autarkic and dictator regimes of the savings stage, defined in Eq.* (B.17) *and* (B.16)*, satisfy the following:*

1. *If $\gamma < 1$ (high intertemporal elasticity of substitution), then $MPC^{aut} < MPC^{dict}$ and MPC^{dict} is decreasing in α, i.e. the marginal propensity to consume decreases as the parent becomes more altruistic.*

2. *If $\gamma = 1$ (unit elasticity), then $MPC^{aut} = MPC^{dict}$ and $MPC^{dict} = 1/(1+\beta)$ is independent of the degree of altruism, α.*

3. *If $\gamma > 1$ (low intertemporal elasticity), then $MPC^{aut} > MPC^{dict}$ and MPC^{dict} is increasing in α, i.e. the marginal propensity to consume* ***increases*** *as the parent becomes more altruistic.*

Proof. First, note from Eq. (B.18) that i) $\tilde{R} < R$ for any $\alpha > 0$ and $\gamma > 0$, and ii) $\tilde{R}$ is increasing in α for any $\gamma > 0$. The comparative-statics results in the proposition then follow directly from Eq. (B.17) and (B.16). ■

Initial gifts. The strategy to find initial-period gifts is as follows. We first find the parent's optimal gift within each regime (constrained, dictator and autarky), which can be done using first-order conditions since all problems are concave. We can then compare the values coming from the three sub-problems to find the global solution.

Recall here that to maximize a concave function on an interval, we have to pick (i) the lower boundary if the unconstrained maximizer falls to the left of the feasible set (this can mean either giving zero gifts or staying at the left corner of the region under consideration), (ii) the unconstrained solution if the unconstrained maximizer falls inside the feasible interval, (iii) the upper boundary if the unconstrained maximizer falls to the right of the feasible set. Mathematically, this yields the somewhat ugly, yet closed-form, expressions

$$\tilde{A}^{cd}_{0,s}(a) = \arg \max_{A \in [a, \min\{x^{cd}_{0,s}, a+y^p_0\}]} K(A;a) \tag{B.21}$$

$$= \max\left\{a, \min\left\{A^{cd}_{0,s}(a), x^{cd}_{0,s}\right\}\right\} \qquad \text{defined for } a < x^{cd}_{0,s}, \tag{B.22}$$

$$\tilde{A}^{dict}_{0,s}(a) = \arg \max_{A \in [\max\{a, x^{cd}_{0,s}\}, \min\{x^{aut}_{0,s}, a+y^p_0\}]} K(A;a) \tag{B.23}$$

$$= \max\left\{a, x^{cd}_{0,s}, \min\left\{A^{dict}_{0,s}(a), x^{aut}_{0,s}\right\}\right\} \qquad \text{defined for } x^{cd}_{0,s} - y^p_0 < a < x^{aut}_{0,s}, \tag{B.24}$$

$$\tilde{A}^{aut}_{0,s}(a) = \arg \max_{A \in [\max\{a, x^{aut}_{0,s}\}, a+y^p_0]} K(A;a) \tag{B.25}$$

$$= \max\left\{a, x^{aut}_{0,s}, A^{aut}_{0,s}(a)\right\} \qquad \text{defined for } x^{aut}_{0,s} - y^p_0 < a, \tag{B.26}$$

where the parent's payoff function K is given by Eq. (15), $A^{cd}_{0,s}(a)$ by Eq. (B.12), $A^{dict}_{0,s}(a)$ by Eq. (B.6), $A^{aut}_{0,s}(a)$ by Eq. (B.10), $x^{cd}_{0,s}$ by Eq. (B.19), and $x^{aut}_{0,s}$ satisfies Eq. (B.20). We note that the corner solution that the parent gives all of $a + y^p_0$ to the child and consumes zero is irrelevant since the Inada condition on $u(\cdot)$ ensures that this is sub-optimal.

Switch from constrained to dictator regime. Now, we can find the cut-off $x^{cd}_{0,g}$ where the dictator-savings region starts to dominate the constrained region by solving

$$\underbrace{K(\tilde{A}_{0,s}^{cd}(a);a) - K(\tilde{A}_{0,s}^{dict}(a);a)}_{\equiv X_{cd}(a)} = 0 \qquad \text{for } a \in [\max\{x_{0,s}^{cd} - y_0^p, 0\}, x_{0,s}^{cd}]. \tag{B.27}$$

Note that the function $X_{cd}(\cdot)$ is continuous since it is the sum of two continuous functions. Importantly, the function X_{cd} can cross zero only once on the relevant range, as is implied by Lemma 5.

If at $a = 0$ a shot to the dictator-savings region is already feasible, it may be that (i) Eq.(B.27) is already negative (or zero) at $a = 0$, which should be ruled out before solving. Also, it may be that (ii) Eq.(B.27) is zero at the upper end $a = x_{0,s}^{cd}$, in which case this is the switch to the dictator-savings region. If none of (i) or (ii) is true, then $x_{0,g}^{cd}$ can be found by finding the unique root of $X_{cd}(a)$ on the range $[\max\{x_{0,s}^{cd} - y^p, 0\}, x_{0,s}^{cd}]$.

Switch to autarkic regime. Denote the parent's value from the better out of regimes *cd* and *dict* as

$$P_{0,g}^{low}(a) = \mathbb{I}(a < x_{0,g}^{cd})K(\tilde{A}_{0,s}^{cd}(a);a) + \mathbb{I}(a \geq x_{0,g}^{cd})K(\tilde{A}_{0,s}^{dict}(a);a), \tag{B.28}$$

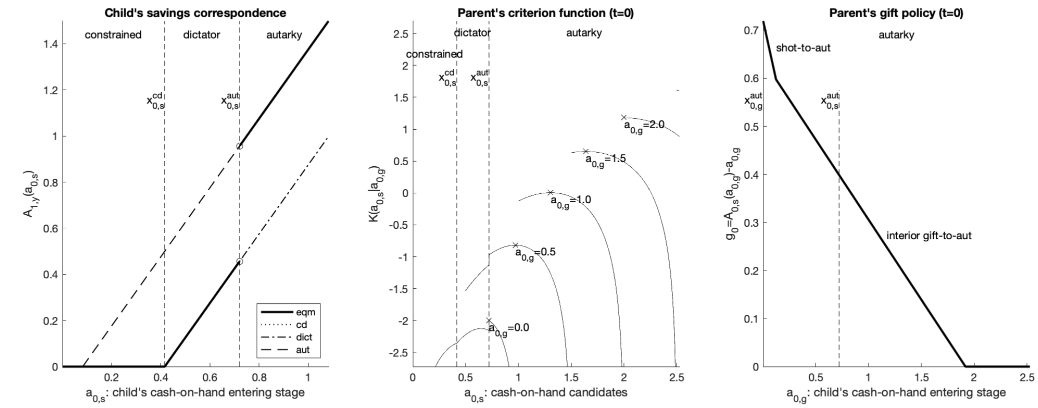
which is well-defined for $a \in [0, x_{0,s}^{aut}]$ (i.e. up to the point where only autarky is a possibility) and continuous.

Finally, we can find the cut off where the switch from the "low" regime (*cd* or *dict*) to autarky takes place. The solution is the number $x_{0,g}^{aut}$ that solves

$$\underbrace{P_{0,g}^{low}(a) - K(\tilde{A}_{0,s}^{aut}(a);a)}_{\equiv X_{aut}(a)} = 0 \qquad \text{for } a \in [\max\{x_{0,s}^{aut} - y_0^p, 0\}, x_{0,s}^{aut}]. \tag{B.29}$$

Again, there can only be one solution by Lemma 5 – once autarky becomes optimal, the optimum cannot jump back to the low regime. Since $X_{aut}(\cdot)$ is the sum of two continuous functions, it is also continuous itself. If at $a = 0$ a shot to autarky is already feasible, it may be that (i) X_{aut} is already negative (or zero) at $a = 0$, which should be ruled out before solving – in this case autarky is *always* played.[38] Note that by Prop. 2.2, X_{aut} must switch to negative before reaching the upper end $a = x_{0,g}^{cd}$, i.e. shots to autarky will be optimal once we get close enough to the autarky threshold since the parent's continuation value has an upward jump discontinuity. Thus, if case (i) does not apply, we find $x_{0,g}^{aut}$ by finding the root of $X_{aut}(a)$ on $[\max\{x_{0,s}^{aut} - y_0^p, 0\}, x_{0,s}^{aut}]$.

Figure C.1: College case (selected outcomes at $t = 0$)



Parameterization as in baseline , but interest rate increased to $R = 3$. $u(c) = \ln(c)$, $\alpha = \beta = y_0^p = y_1^p = 1, y_1^k = 1/4$.

C Additional results

Figure C.1 shows selected outcomes for the college case discussed in Section 3.1. Figure C.2 complements Section 4.2; it shows the effect of increasing the child's expected final-period endowment to the alternative value $\mathbb{E}(y_1^k) = 1.5$, keeping the other parameters as in Fig. 6.

D Calibration of earnings process in numerical exercise

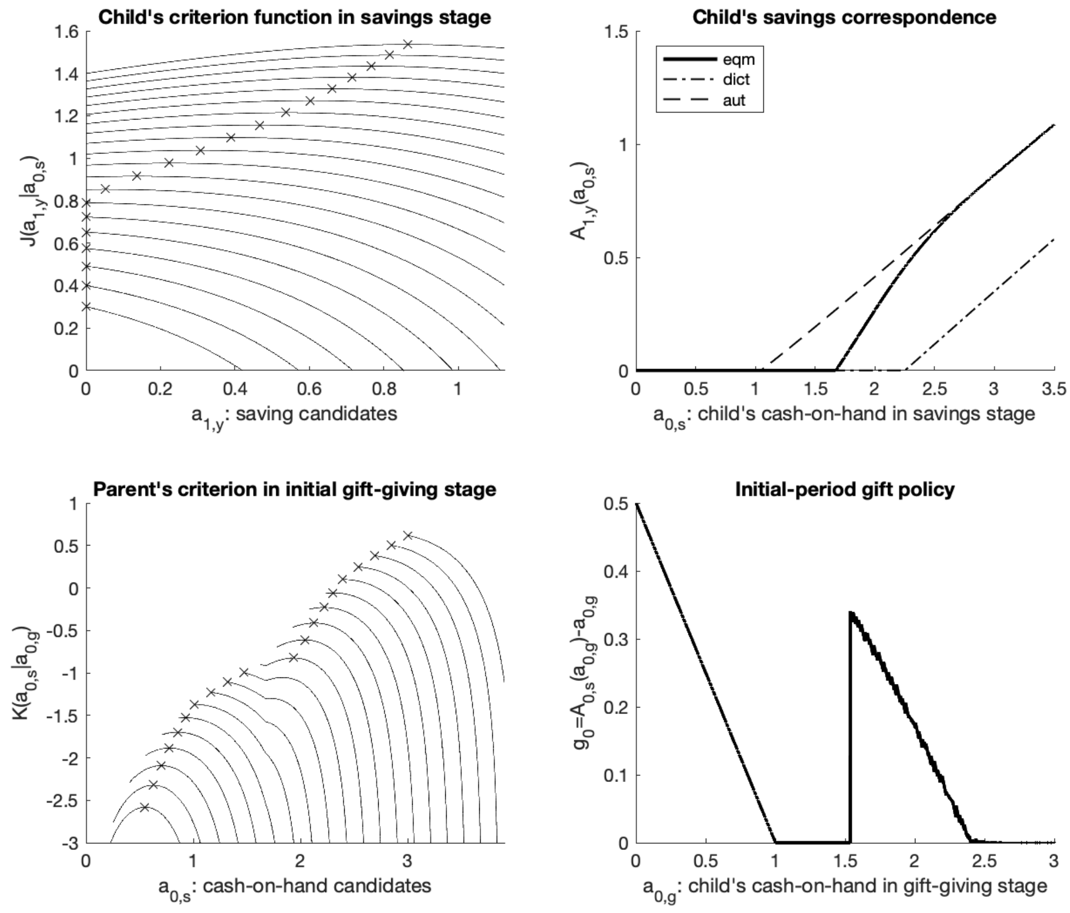
Our aim in this section is to construct an income process across generations that is quantitatively consistent (i) the high persistence of earnings, (ii) the correlation of children's earnings with those of their parents, and (iii) the hump shape of earnings over the life cycle documented in the literature. We then use U.S. data to discipline the process.

Setup. To formalize these three key features we consider a three-period overlapping-generations structure with three generations alive: (y) young of working age (25-45), (o) old of working age (45-65), and (r) retired (65-85).[39] Time is doubly infinite, $t \in \{\dots, -1, 0, 1, \dots\}$, with one time unit corresponding to 20 years. We denote by y_t^i, $i \in \{y, o, r\}$, the logarithm of net labor income of generation i at time t, where y_t^r represents Social Security income. We assume that the logarithm of net labor income is the sum of a deterministic Mincer profile over the lifecycle and a persistent

[38]Note that if the lower bound is $x_{0,s}^{aut} - y_0^p$, the parent would never prefer this lower bound by the Inada condition since parent consumption is zero in this case.

[39]We note that this structure is merely a tool to put discipline on the income process; it differs from the generational structure of our theoretical model.

Figure C.2: Continuous-support shock and $\mathbb{E}(y_1^k) = 1.5$ (selected outcomes at $t = 0$)



y_1^k follows log-normal distribution with expected value 1.5 and standard deviation 1. Simulation grid size: $N = 5,000$. Parameters: $u(c) = \ln(c)$, $\alpha = \beta = R = 1$, $y_0^p = y_1^p = 1$.

component[40]

$$y_t^i = \bar{y}^i + x_t^i, \quad \forall t, \quad \text{where } \mathbb{E}(x_t^i) = 0 \, \forall t, i.$$

Specifically, we will determine $\bar{y}^i$ from the Mincerian age profile of earnings in age, capturing the hump-shaped nature of earnings over the lifecycle. In line with literature, we assume that the persistent component for own earnings follows a random walk,

$$x_t^o = x_{t-1}^y + \epsilon_t^o, \quad \epsilon_t^o \sim \mathcal{N}(0, \sigma_o^2), \tag{D.1}$$

where ϵ_t^o is the innovation to the permanent component. To capture the correlation between parents' and children's earnings in the data, we assume that

$$x_t^y = \beta_y x_t^o + \epsilon_t^y, \quad \epsilon_t^y \sim \mathcal{N}(0, \sigma_y^2), \tag{D.2}$$

where β_y is the intergenerational elasticity of child's long-run earnings with respect to parent's long-run earnings, and ϵ_t^y is the innovation to the inherited component.[41] The shocks ϵ_t^y and ϵ_t^o are assumed to be independent from each other and across time. For retirement income, we assume that deviations from the cohort mean are the same in percentage terms as for old-working-age earnings, i.e.

$$x_t^r = x_{t-1}^o. \tag{D.3}$$

Stationary variances. Under the above assumptions, the joint process for earnings is stationary and the variables $\{x_t^y, x_t^o, x_t^r\}$ are jointly normally distributed, having constant (unconditional) variance. We denote by $\sigma_{x^i}^2 = Var(x_t^i)$ the stationary variance of earnings for $i \in \{y, o, r\}$. From the laws of motion (D.1) and (D.2) it then follows that these variances satisfy

$$\sigma_{x^y}^2 - \beta_y^2 \sigma_{x^o}^2 = \sigma_y^2, \tag{D.4}$$

$$\sigma_{x^o}^2 - \sigma_{x^y}^2 = \sigma_o^2. \tag{D.5}$$

Calibration. First, we set the intergenerational elasticity of earnings to $\beta_y = 0.5$ following Chetty et al. (2014). We follow a common assumption in the literature that the permanent component of individual log earnings, here: $\{x_t^y, x_{t+1}^o\}$, follows a random walk over the life cycle. This implies

[40] We omit the commonly assumed transitory component since short-term (e.g. yearly) fluctuations will tend to average out over one model period of 20 years.

[41] Note here that typical estimates of intergenerational elasticities of earnings usually estimate long-run earnings *late* in life for parents and *early* in life for children due to data limitations. Thus our specification here, which ignores parents' early and children's late earnings, should not be too far off.

that the variance of individual earnings increases linearly in time. Since our time periods are of equal length, this motivates the following assumption:

$$\sigma^2_{x^y} = 0.5\sigma^2_{x^o}. \tag{D.6}$$

To determine the variance and the age-varying means of earnings, we run a quadratic Mincer regression using 2010 U.S. census data without any other controls (since we want the process to include all sources of heterogeneity in earnings). We restrict the sample to males with pre-tax wage and salary income of at least \$20K. To get a rough approximation for after-tax income we employ a 20% proportional tax rate. We set $\bar{y}^y = 10.65$ ($\bar{y}^o = 10.77$) equal to the average of predicted log net labor income over the ages 25-44 (45-64) and calculate an estimate $\sigma^2_{x^o} = 0.65$ as the mean-squared error from a Mincer regression based on ages 45-64.[42] (D.4), (D.5) and (D.6) then imply that the shock variances can be obtained as

$$\sigma^2_y = \sigma^2_{x^o}(0.5 - \beta^2_y) \tag{D.7}$$

$$\sigma^2_o = 0.5\sigma^2_{x^o}. \tag{D.8}$$

As for retirement income, we assume that there is a uniform net replacement rate κ, which in logarithms translates to

$$y^r_t = \ln(\kappa) + y^o_{t-1}.$$

Using (D.3) and taking expectations we get

$$\bar{y}^r = \ln\kappa + \bar{y}^o,$$

where we use the net replacement rate $\kappa = 0.5$ reported by the OECD for the U.S.

Drawing from the process. We draw income realizations from this process as follows. First, we draw $N = 10^5$ realizations of x^o_0 realizations from $\mathcal{N}(0, \sigma_{x^o})$. Next, we draw ϵ^y_0 realizations from $\mathcal{N}(0, \sigma_y)$. Taken together, we then obtain the child's income for the initial period using $y^y_0 = \bar{y}^y + x^y_0 = \bar{y}^y + \beta_y x^o_0 + \epsilon^y_0$, the parent's income in the initial period being $y^o_0 = \bar{y}^o + x^o_0$. To obtain the income for the child in the final period, we draw ϵ^o_1 from $\mathcal{N}(0, \sigma_o)$. Children's final-period income is then given by $y^o_1 = \bar{y}^o + x^y_0 + \epsilon^o_1$. We then calculate parent permanent income as $\frac{R}{1+R}(y^o_0 + R^{-1}y^r_1)$ and finally divide the child's endowments by parent's permanent income, so that y^k_0 and y^k_1 are relative to parent permanent income and $y^p_0 = y^p_1 = 1$.

[42] We prefer to impose the random-walk assumption (D.6) to obtaining an estimate for $\sigma^2_{x^y}$ from the Mincer regression directly since the earnings data are not stationary in recent times, as earnings inequality has increased over time. Hence, young workers' earnings likely display much more variance than in the past.