Blast from the past: The altruism model is richer than you think

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This paper revisits the standard two-period altruism model along the lines of Altonji, Hayashi, and Kotlikoff (1997). We provide the first full theoretical characterization of the model in a deterministic setting, with (near-)closed-form solutions for the power-utility case. We show that policy and value functions are discontinuous. There exists a range of initial conditions under which the altruistic donor gives an initial-period gift that enables the recipient to stay in autarky for both periods. Our results revise important predictions that have been used to test the altruism model, such as on the increasingness of transfers in the recipient's income and on the transfer-income derivative. The results are robust to introducing income shocks and to varying assumptions on parent savings.

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1 Introduction

A popular version of the standard altruism[1] model is studied by Altonji et al. (1997) (AHK). An altruistic parent can provide financial transfers to her child in a two-period setting with savings. Commitment is absent and thus strategic considerations arise in the decision process. The timing is set up as a standard Stackelberg game where the parent is the leader and the child the follower. There are two main reasons for its popularity in applied work. First, the sequential timing of decisions yields a unique outcome,which is unlike in a simultaneous-move game where there can be multiple equilibria.[2] Second, AHK provide a proof that the model with uncertainty over the child's income and liquidity constraints pins down the timing of transfers, arguing that transfers in the first period can only flow if the child is liquidity-constrained. Their proof is based on first-order conditions and (implicitly) assumes continuous policies. As a result, it is commonly believed that standard optimization techniques that are based on first-order conditions can be used to find a solution to multi-period altruism models.

However, we show in this paper that value and policy functions have multiple kinks (i.e. non-differentiabilities) and even discontinuities, which invalidates the use of first-order conditions. We provide a full theoretical characterization of the model in a deterministic setting. Our baseline model is a simplified version of AHK, in which we strip out the parent's savings decision and the uncertainty over the child's income. This simplifies the analysis substantially but leaves intact all qualitative model features, which we show by extending the baseline model. Interestingly, eliminating the savings choice of the parent *does not* remove strategic considerations for the parent, which is what some of the literature had conjectured.

In line with what is commonly believed, we find that there is a unique equilibrium that obtains by backward induction.[3] A technical complication is that at each stage the number of kinks and discontinuities increases. The non-negativity constraint on parent gifts and the borrowing constraint for the child introduce kinks in policy functions. In one-player dynamic-programming problems, these kinks are unproblematic since the Envelope Theorem guarantees that value functions remain smooth despite the kink in the policy function. Economically, the decision maker is indifferent how the marginal unit of assets is allocated. In a game, however, the *second* player has conflicting interests and is usually *not* indifferent on how the marginal unit is spent, introducing kinks in her value function. When going back one stage in the game, these kinks then translate into discontinuities in the second player's policy, which in turn leads to a jump discontinuity in the value function

[1]By *altruism*, we mean preferences of the kind in Becker (1974): The altruists well-being depends on the well-being of someone else.

[2]See Lindbeck & Weibull (1988), who show that multiple equilibria arise in a two-period simultaneous-move altruism model for a set of initial conditions (of positive measure).

[3]Multiple best responses exist at some points in the state space where agents are indifferent, but the set of these points has zero measure in the state space.

of the first player.

More concretely, we show that under weak assumptions on the felicity function, the parent's gift policy in the final period has a kink at the point where gifts turn positive. This induces an upward jump in the child's savings correspondence in the initial period. The child is either i) constrained, ii) saves while anticipating a future transfer (what we term *dictator* savings), or iii) saves knowing that it will not receive a transfer in the final period (*autarkic* savings). The upward jump in savings occurs at the point where the child switches to autarkic savings. Importantly, at this very point the child's policy creates an upward discontinuity in the parent's value function. That is, whereas value-matching holds for the child (it is indifferent between two global optima) it is violated for the parent: the parent strictly prefers the autarkic allocation, since it implies lower transfer efforts in the future. Stepping backwards prior to the child's savings choice, the parent can exploit this large increase in value by nudging the child into autarkic savings by a transfer, which we call a *shot to autarky*. Moreover, we find that for some parameterizations a similar lift occurs for lower child wealth, in which the parent lifts the child from being constrained into the region where it engages in dictator savings (if it is too costly or impossible to lift the kid to autarky).

An important novelty we find –and a caveat to previous literature– is that only *one* type of transfers (*spoon-feeding*) is characterized by the familiar first-order condition (FOC) known from static altruism models. This FOC says that the parent should increase her gift (and increase the child's consumption) as long her own marginal utility of consumption is lower than the marginal utility that she obtains from the child's increased consumption (in the same period). However, shots to autarky follow a profoundly different logic: When lifting the child to autarky, the parent actually induces the child to consume *less* –and save more– than it would have in the absence of the transfer. This is critical since a lot of the empirical literature has, explicitly or implicitly, relied on the the spoon-feeding FOC for testing the altruism model. Our results revise three important predictions of the altruism model: The properly-solved model implies that i) parent transfers are non-monotonic in the child's wealth, ii) parent transfers may induce lower rather than higher contemporaneous child consumption, and iii) the transfer-derivative restriction tested by AHK is not a global but a local property of the model (see the literature below for more on this).

As for the timing of transfers, we find that all configurations are possible. First, transfers may flow in the first but not the second period,which is the case of shots to autarky. Second, transfers may flow in both periods. If a shot-to-autarky does not already occur at the lowest level of the child's starting wealth, then there must be a region where the parent provides transfers and the child is constrained (*spoon-feeding*). In this case, transfers flow in both periods and the child chooses zero savings. These are the transfers identified by AHK and others. In addition, however, we show that there can be another novel type of transfer which lifts the child from the constrained to the dictator-savings region, i.e. transfers flow in both periods while the child is saving.[4] Third, transfers may be zero in the first period but positive in the second period, leading to the well-known Samaritan's Dilemma in which the child chooses her savings inefficiently low. Fourth, the child may be in autarky in both periods.

When felicity is of the power-utility form, we provide closed-form solutions for policies within regimes, which we show to be piecewise linear. The cut-off values, which arise from value-matching conditions, are given either in closed or in implicit form and can be easily obtained numerically using a root-finding algorithm. These closed forms are useful for future literature since they provide a benchmark for evaluating the precision of computational algorithms.

Finally, we show that our main results are robust to changes in the environment. We first introduce uncertainty by assuming that the child is subject to income shocks. Intuitively, when these shocks have continuous support payoff functions become differentiable; they remain non-concave, however. Economically speaking, the convexity in the child's value function is reminiscent of "gambling for resurrection", i.e. risk-lovingness in models with limited liability or bail-outs. When the child has low assets, downside risk is insured by transfers from the parent. On the upside, however, the child enjoys all gains from savings herself when having high wealth. We find that even when assuming income risk that is unrealistically large, the convexity remains and the discontinuities in policies and value functions arise just as in the deterministic case. If income shocks have discrete support, no smoothing occurs at all; to the contrary, the number of kinks and discontinuities will usually increase. Finally, letting also the parent engage in savings leaves our main results unchanged.

Literature review Our paper contributes to theoretical, empirical, and applied literatures.

Theoretically it relates most closely to Lindbeck & Weibull (1988) and Bruce & Waldman (1990). Lindbeck and Weibull study a deterministic two-period simultaneous-move game. The simultaneous-move feature leads to multiple equilibria on a subset of the state space, one outcome being efficient and the other featuring over-consumption by the child. This occurs for intermediate child assets, as is the case for our shots to autarky. While the simultaneous-move setting delivers no clear prediction which regime is played in this region, in our sequential-move setup the parent uses her first-mover advantage to nudge the economy into the autarkic regime, which the parent prefers. But, as we show, other important complications surface under sequential decision-making. Bruce and Waldman study sequential decisions in a deterministic two-period model. As part of the discussion they conjecture that there is an equilibrium where first-period transfers lead to efficient savings by the recipient.[5] In contrast, we actually show that shots-to-autarky occur (under weak

[4]Furthermore, there can be transfers within the dictator-savings region (local transfers which do not alter the regime, i.e. the economy stays in the dictator-savings region).

[5]Their arguments rely on continuity of policies and first-order conditions, hence they may only characterize a subset of equilibria.

assumptions) and characterize them. As mentioned already, an important caveat that our paper raises with respect to this early literature is that the *operativeness* of transfer motives (i.e. transfers being positive) is not equivalent the parent's spoon-feeding FOC holding.

In our own work, we have studied altruism models in continuous time. In most of this work (Barczyk & Kredler, 2014*a*, Barczyk, 2016, Barczyk & Kredler, 2018), we use Brownian shocks that together with the continuous-time assumption are sufficient to smooth value functions such that all transfers are of the spoon-feeding type. In Barczyk & Kredler (2014*b*) we study a deterministic infinite-horizon game in continuous-time, which is most similar to this paper. We find that when restricting attention to Markovian strategies, no shots to autarky can occur. The two results are compatible since shots to autarky are non-Markovian in nature: the parent provides such transfer only in the initial but not the final period. Furthermore, Barczyk & Kredler (2014*b*) find an equilibrium an which both players pool their wealth in the long run; this type of equilibrium does not obtain in this two-period setting since the horizon is finite.

A sizeable literature attempts to empirically disentangle motives for financial transfers among family members. Cox (1987) and Cox & Rank (1992), for example, argue their data is more consistent with exchange than altruism. However, they rely on the monotonicity of altruistic transfers in recipient income, which we show to be violated. Closely related is the empirical test of the transfer-income derivative by AHK (1997). We show that this transfer-income restriction holds at most *locally* within a region, but fails to hold globally. AHK provide a proof that parents delay transfers as long as possible and that all transfers are of the spoon-feeding type. However, their proof implicitly assumes continuity and differentiability of policy functions, and is thus only locally valid. Of course, it may still be the case that the transfer-income restriction holds at least "approximately" in a realistically-calibrated model. However, recent work by Chu (2019) suggests otherwise. She solves a life-cycle model with two altruistic agents numerically using brute-force grid search and finds that shots to autarky give rise to a transfer-income derivative far below one. We view her numerical results as complementary to the theoretical approach we take here.

A recent literature has embedded altruistic players into larger quantitative models. Kaplan (2012) studies the role of altruistic parents in insuring their children against labor-market risk by providing the possibility to move back home. In order to simplify strategic considerations he assumes that parents cannot save. But, we find that the complications arising from strategic interactions remain even if the parent cannot save. Boar (2020) studies the importance of savings by altruistic parents to insure children against labor-income risk by providing inter-vivos transfers, focusing on equilibria with spoon-feeding transfers. Our paper contributes to this literature by guiding the quest for appropriate algorithms to solve these discrete-time models with altruism.

The remaining paper is structured as follows. Section 2 provides the theoretical analysis of the benchmark model; most of the proofs are relegated to Appendix A. In Section 3, we study different types of equilibria arising in our framework, which are interesting in their own right. In Section 4 we extend the baseline model by (1) allowing the parent to save, and (2) by introducing uncertainty over the child's income. Section 5 concludes.

2 An off-the-shelf model

We study a two-period deterministic model of a parent and a child. The two periods are denoted by $t = 0, 1$. In each period there are three stages. We refer to them as income (y), gift-giving (g), and savings (s) stage, respectively. In the first stage, the child receives income y_t^k and the parent receives income y_t^p. In the second stage, the parent decides on a non-negative transfer g_t to the child; the parent then consumes what is left and obtains utility from it. The third stage is the savings stage in which the kid decides how much to save in a risk-free asset that pays a gross return $R \geq 1$; the child is subject to a no-borrowing constraint, i.e. we require $a_{t+1,y} \geq 0$.

Preferences of the child are defined over the child's current- and future-period consumption (c_0^k, c_1^k) and represented by $u(c_0^k) + \beta u(c_1^k)$, where $\beta > 0$. The preferences of the parent include its own current- and future-period consumption (c_0^p, c_1^p) and also the consumption allocation of the child, $u(c_0^p) + \beta u(c_1^p) + \alpha[u(c_0^k) + \beta u(c_1^k)]$, where $\alpha > 0$ measures the strength of the parent's altruism towards the child. We make the following assumptions on the felicity function, which are standard and rather weak:

Assumption 1. $u(\cdot)$ *is twice continuously differentiable with* $u'(c) > 0$ *and* $u''(c) < 0$ *for all* c *and satisfies the Inada conditions* $\lim_{c\to 0} u'(c) = \infty$ *and* $\lim_{c\to\infty} u'(c) = 0$.

In order to study a non-trivial environment we will usually make use of the following condition:

Condition 1 (Gifts possible in final stage)**.** $\alpha u'(y_1^k) > u'(y_1^p)$.

It ensures that transfers are possible in the final period, which is the case if the parent wants to give to the child when the child has not saved anything. If this condition is violated, autarky is the only outcome.

The state variable of the game is the child's *cash-on-hand* coming into each stage. Specifically, when entering the income stage at time t, we denote cash-on-hand by $a_{t,y}$. We will treat the assets $a_{g,y}$ with which the child enters the game as a parameter of our model. When entering the gift-giving stage, cash-on-hand is $a_{t,g} = a_{t,y} + y_t^k$. The parent takes $a_{t,g}$ as given and chooses child's cash-on-hand $a_{t,s} \geq a_{t,g}$, or expressed in gifts, $g_t = a_{t,s} - a_{t,g} \geq 0$. At the beginning of the savings stage the child's cash-on-hand is $a_{t,s} = a_{t,g} + g_t$.

To characterize the solution, we will make use of stage-contingent value functions. Let $V_{t,i}(a)$ be the child's value and $P_{t,i}(a)$ the parent's value when child's cash-on-hand is a coming into stage $i \in \{y, g, s\}$ of period $t \in \{0, 1\}$. In general, we can think of player's actions (gifts and savings) as setting cash-on-hand for the next stage of the game. We will denote the parent's cash-on-hand policy by $A_{t,s}(a_{t,g})$ and the child's consumption-savings policy by $A_{t+1,y}(a_{t,s})$.

We will illustrate our results with numerical examples. These are computed using the closed-form solutions that we derive for the power-utility case in Appendix B. That is, these examples invoke additionally

Assumption 2 (Power utility)**.** *Utility is of the power form, i.e.* $u(c) = c^{1-\gamma}/(1-\gamma)$*, where* $\gamma > 0$ *and where we define* $u(c) = \ln c$ *for the case* $\gamma = 1$.

2.1 Final period

We solve the game by backward induction. Obviously, in the savings stage of the final period the child's optimal policy is to leave no resources behind and the policy and value functions are given by

$$A_{2,y}(a_{1,s}) = 0, \qquad V_{1,s}(a_{1,s}) = u(a_{1,s}), \qquad P_{1,s}(a_{1,s}) = \alpha u(a_{1,s}). \tag{1}$$

In order to keep track of the smoothness properties of value and policy functions, we state the following obvious result:

Lemma 1 (Smoothness in final stage)**.** *Under Ass. 1, value functions and the policy function in the final-period consumption stage are twice continuously differentiable.*

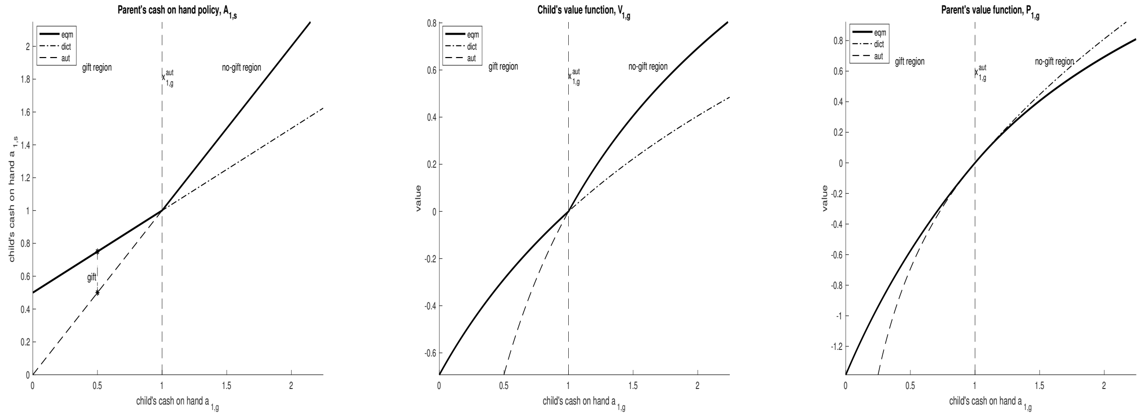
Going back to the gift-giving stage of the final period, the parent's problem is then given by

$$P_{1,g}(a_{1,g}) = \max_{a_{1,s} \in [a_{1,g}, a_{1,g} + y_1^p]} \left\{ u(y_1^p + a_{1,g} - a_{1,s}) + P_{1,s}(a_{1,s}) \right\}. \tag{2}$$

The parent chooses child's next-stage cash-on-hand, $a_{1,s}$. The lower bound of the feasible set, $a_{1,g}$, says that the parent must leave the child with at least as much as when entering the stage, which is nothing but the non-negativity constraint on gifts. The upper bound of the feasible set equals total family resources, $a_{1,g} + y_1^p$, i.e. the maximal transfer the parent can give is its income y_1^p. Combining Lemma 1 and Ass. 1, we see that the maximization problem 2 is well-behaved, i.e. the parent maximizes a concave criterion on a convex set.

In general, our strategy for solving the game will be as follows. We define auxiliary problems in each stage that describe the solution in a particular regime; for example, a regime in the final-period gift-giving stage is if gifts flow or not. We then characterize the value and policy functions in these well-behaved auxiliary problems and piece them together to find the solution to the game.

Figure 1: $t = 1$: Gift-giving stage



Equilibrium and auxiliary outcomes in gift-giving stage of the final period. (a) Parent's cash-on-hand policy, (b) child's value function, and (c) parent's value function. Gifts flow below threshold value $x_{1,g}^{aut} = 1$. Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and $y^k = 1/4$.

In the final-period gift-giving stage, the first regime we define is the *dictator* (dict) environment in which we give the parent the power over all the family's resources, i.e. we allow the parent to choose positive as well as negative gifts:

$$P_{1,g}^{dict}(a_{1,g}) = \max_{a_{1,s} \in [0, a_{1,g} + y_1^p]} \left\{ u(y_1^p + a_{1,g} - a_{1,s}) + P_{1,s}(a_{1,s}) \right\}. \tag{3}$$

This problem differs from the true problem (2) only in that it enlarges the feasible set. It is easy to see that concavity and the Inada condition from Ass. (1) guarantee a unique interior solution to the dictator problem. This solution is implicitly defined from the parent's first-order condition

$$u'\big(\underbrace{y_1^p + a_{1,g} - A_{1,s}^{dict}(a_{1,g})}_{=c_1^p}\big) = \alpha u'\big(\underbrace{A_{1,s}^{dict}(a_{1,g})}_{=c_1^k}\big). \tag{4}$$

This is the familiar first-order condition for altruistic transfers, which says that the parent equates marginal utility from own consumption to the marginal utility from child consumption (weighted by the strength of altruism) – equation (4) is what is commonly meant by *operative* transfer motive.

The second auxiliary problem we define in this stage is *autarky*, i.e. an environment in which we force gifts to be zero. The policy and value functions in this environment are given by

$$A_{1,s}^{aut}(a_{1,g}) = a_{1,g}, \qquad V_{1,s}^{aut}(a_{1,g}) = u(a_{1,g}), \qquad P_{1,s}^{aut}(a_{1,g}) = u(y_1^p) + \alpha u(a_{1,g}). \tag{5}$$

Again, the policy and value functions of the autarky environment are shown in Fig. 1 for our example.

Armed with these two auxiliary problems, we now return to the parent's true gift-giving problem, (2).[6] It is easy to see that the parent's optimal gift policy is given by the maximum of the two auxiliary policies, since the child must always have at least what it has under autarky:

$$A_{1,s}(a_{1,g}) = \max\left\{A_{1,s}^{dict}(a_{1,g}), A_{1,s}^{aut}(a_{1,g})\right\} = \max\left\{A_{1,s}^{dict}(a_{1,g}), a_{1,g}\right\}. \tag{6}$$

Fig. 1 shows the policy and the associated value functions for an example with logarithmic utility; since utility is homothetic, the parent always chooses the same split of resources and the optimal policy is thus linear.

The parent chooses a positive gift when the child is poor, but then switches to autarky once the child is rich enough (the parent would want to take away from the child, but cannot). We can characterize the threshold $x_{1,g}^{aut}$ at which the regime changes from dictator to autarky by

$$u'(y_1^p) = \alpha u'(x_{1,g}^{aut}), \tag{7}$$

which we note to be always well-defined under Ass. 1. Note that Cond. 1 implies $x_{1,g}^{aut} > y_1^k$, which means that the child will receive gifts if her savings are low enough. If Cond. 1 does not hold, however, the child never receives gifts in the final stage.

It is worthwhile to observe in Fig. 1 what the regime change implies for the smoothness of the value functions. Since the parent equates the marginal utility from consuming and from giving the first dollar in gifts at $x_{1,g}^{aut}$, the parent's value function is differentiable (by the Envelope Theorem). The child, however, cares only about her own consumption. Hence the kink in the parent's gift-giving policy (in the left panel) directly translates into a kink in the child's value (in the middle panel).

In the following Lemma we summarize the most important features of the gift-giving stage in the final period (for the proof see Appendix A).

Lemma 2 (Final period: kinks in gift-giving stage)**.** *Suppose that Ass. 1 holds and let $x_{1,g}^{aut} > 0$ be defined by Eq. (7). Then the parent's policy at $t = 1$ is to give positive gifts for $a_{1,g} < x_{1,g}^{aut}$, but no gifts for $a_{1,g} \geq x_{1,g}^{aut}$. The parent's policy function $A_{1,s}(a_{1,g})$ is continuously differentiable everywhere except for the point $x_{1,g}^{aut}$, satisfying*

$$A'_{1,s}(a_{1,g}) \begin{cases} \in (0,1) & \textit{for } a_{1,g} < x_{1,g}^{aut}, \\ = 1 & \textit{for } a_{1,g} > x_{1,g}^{aut}, \end{cases}, \tag{8}$$

$$A'^{-}_{1,s}(x_{1,g}^{aut}) < A'^{+}_{1,s}(x_{1,g}^{aut}) = 1, \tag{9}$$

[6] We discuss the problem here somewhat informally, but provide a formal statement and proof in Lemma 2.

i.e. there is a downward kink in the policy function between the two regions.[7] *The child's value function $V_{1,g}(\cdot)$ has the same smoothness profile as the policy function, i.e. it is continuously differentiable except for a downward kink at $x_{1,g}^{aut}$. However, the parent's value function $P_{1,g}(\cdot)$ is continuously differentiable everywhere with*

$$P'_{1,g}(a_{1,g}) = \alpha u'\big(A_{1,s}(a_{1,g})\big). \tag{10}$$

The parent's value function $P_{1,g}(\cdot)$ is (globally) strictly concave; the child's value function $V_{1,g}(\cdot)$ is strictly concave on the range $(x_{1,g}^{aut}, \infty)$.

We note here the following. Cond. 1 is not required for the characterization in this Lemma. However, if Cond. 1 does not hold, the dictator/gift-giving region $a_{1,g} < x_{1,g}^{aut}$ cannot be reached on the equilibrium path. Furthermore, in our example the child's value function is concave on the range $(0, x_{1,g}^{aut})$ because of the linear gift-giving policy. This, however, need not be the case in general since gift-giving policies may be convex for different utility specifications.

2.2 Initial period

2.2.1 Savings stage

Given cash-on-hand $a_{0,s}$, the child's problem in the savings stage of the first period is given by

$$V_{0,s}(a_{0,s}) = \max_{a_{1,y} \in [0, Ra_{0,s}]} J(a_{1,y}; a_{0,s}) \tag{11}$$

where the child's criterion function is given by

$$J(a_{1,y}; a_{0,s}) = u\big(a_{0,s} - a_{1,y}/R\big) + \beta V_{1,y}(a_{1,y}). \tag{12}$$

Note here that the choice variable $a_{1,y}$ is child's cash-on-hand entering the period-1 income stage and $V_{1,y}(a_{1,y}) = V_{1,g}(a_{1,y} + y_1^k)$ is the child's continuation value in that stage. The child chooses $a_{1,y}$ subject to a no-borrowing constraint.

It turns out that the child's problem in (11) is non-standard since the criterion function $J(\cdot, a_{0,s})$ is non-concave, which is directly implied by the downward kink of $V_{1,y}(\cdot)$. Figure 2 shows examples of the child's payoff function for various levels of incoming cash-on-hand $a_{0,s}$ when varying $a_{1,y} \in [0, Ra_{0,s}]$. The most important feature it highlights is the downward kink, which we describe in the following Lemma:

[7] Here, $f^{'-}(x)$ denotes the left and $f^{'+}(x)$ denotes the right derivative of $f(\cdot)$ at x.

Lemma 3 (Properties of child's criterion function $J(\cdot; a_{0,s})$ in savings stage)**.** *Suppose that Ass. 1 holds and define $x^{aut}_{1,y} = x^{aut}_{1,g} - y^k_1$ as the minimal savings that make the child autarkic at $t = 1$. For fixed $a_{0,s}$, the function $J(a'; a_{0,s})$ defined in Eq.* (12) *is continuous in a', concave in a' for $a' \geq x^{aut}_{1,y}$ and differentiable everywhere but in the point $a' = x^{aut}_{1,y}$. If Cond. 1 holds, then $J(\cdot, a_{0,s})$ has a downward kink when entering autarky, i.e. $J'^{-}(x^{aut}_{1,y}; a_{0,s}) < J'^{+}(x^{aut}_{1,y}; a_{0,s})$.*

Proof. Given the definition of $J(\cdot)$ in Eq. (12), the claimed properties follow directly from Ass. 1 and the properties of $V_{1,y}(a) = V_{1,g}(a + y^k_1)$ from Lemma 2. ∎

As is well-known, in the dictator regime the parent decreases gifts as the child saves more, which amounts to a "tax on savings". However, this wedge is not present in the autarkic regime. This becomes visible when taking the first-order condition in (11) that gives us the child's Euler equation:

$$u'(c^k_0) \geq R\beta u'(c^k_1) A'_{1,s}(a_{1,g}), \tag{13}$$

where $a_{1,g} = a_{1,y} + y^k_1$ and which must hold with equality whenever savings are positive. From the properties of $J(\cdot)$, it is clear that this equation is necessary but not sufficient for a solution; furthermore, it holds with inequality when the child is borrowing-constrained. In the Euler equation (13), observe that for savings $a_{1,y}$ such that the child is autarkic, i.e. $a_{1,g} > x^{aut}_{1,g}$, we have $A'_{1,s}(a_{1,g}) = 1$ and thus the standard Euler equation obtains. For savings below the threshold $x^{aut}_{1,g}$, the parent responds by reducing the transfer to the child in the final period, $0 < A'_{1,s}(a_{1,g}) < 1$, which leads to an Euler equation with a distortion. This wedge creates a disincentive for the kid to save.

Now, return to Fig. 2 and observe that the child's criterion has two local maxima for intermediate values of $a_{0,s}$ which is due to the kink in the continuation value. This convexity leads to a discontinuous savings policy with an upward jump. Before stating this result formally, it is first useful to establish that the child's savings policy is weakly increasing:

Lemma 4 (Increasingness of savings correspondence)**.** *Under Ass. 1, the savings correspondence $A_{1,y}(a)$ is increasing in the following sense: If savings A are optimal for some state a, then an optimal savings policy for any higher starting wealth $a + \delta$ must be such that at least A is saved. To be precise, if $A \in A_{1,y}(a)$ for fixed a, then $A - \epsilon \notin A_{1,y}(a + \delta)$ for any $\epsilon \in (0, A]$, for all $\delta > 0$.*

We are now in a position to state the key characteristics of the savings stage in the first period. Under general conditions, it turns out that the child's savings policy must be discontinuous, which in turn leads to a discontinuity in the parent's value function.

Proposition 2.1 (Discontinuous policy and value function in savings stage)**.** *Suppose that Ass. 1 and Cond. 1 hold. Then there exists $x^{aut}_{0,s} \in (0, \infty)$ such that the child chooses savings* *leading into autarky at $t = 1$ for all $a_{0,s} > x^{aut}_{0,s}$, but savings are such that gifts flow at $t = 1$ for $a_{0,s} < x^{aut}_{0,s}$. Optimal savings on the autarky range are characterized by a continuous function. The savings correspondence $A_{1,y}(\cdot)$ is discontinuous (with an upward jump) at $x^{aut}_{0,s}$. The child's value function $V_{0,s}(\cdot)$ is continuous at $x^{aut}_{0,s}$, while the parent's value function $P_{0,s}(\cdot)$ has an upward jump discontinuity at this threshold.*[8]

It is worthwhile to point out that the proof for this proposition invokes the Inada condition $\lim_{c\to\infty} u'(c) = 0$, which implies that the child chooses autarky for high-enough starting wealth. If marginal utility did not vanish (e.g. linear utility), then the child may always prefer to consume today in order to maximize transfers in the final period.

Fig. 2 demonstrates Prop. 2.1. There can be multiple local maximizers in the child's payoff function. A discrete upward jump in the child's optimal savings policy results at the point where there are two global maximizers: The child is just indifferent between autarkic and dictator savings at $x^{aut}_{0,s}$. Moreover, while the child's value function is continuous at this point (since value-matching must hold for the child), this is not the case for the parent. In fact, the parent strictly prefers the autarkic savings allocation at $x^{aut}_{0,s}$ which manifests itself in a discrete upward jump of the parent's value function. The part of the parent payoff that stems from the child's consumption is continuous at this point, since the child is indifferent. However, the parent consumes strictly more in the final period once the regime switches to autarky, which the child does not take into account in her decision.

Finally, we observe that the no-borrowing constraint will usually add additional kinks to the savings policy and to the parent value function. This occurs at $x^{cd}_{0,s}$, which we define as the maximal level of wealth below which the kid is constrained. Similar to the gift-giving stage analyzed before, the child's value function is differentiable at $x^{cd}_{0,s}$ by the Envelope Theorem, but the parent's value function has a kink as she has an additional advantage from child savings – the fact that the parent has to provide fewer transfers.

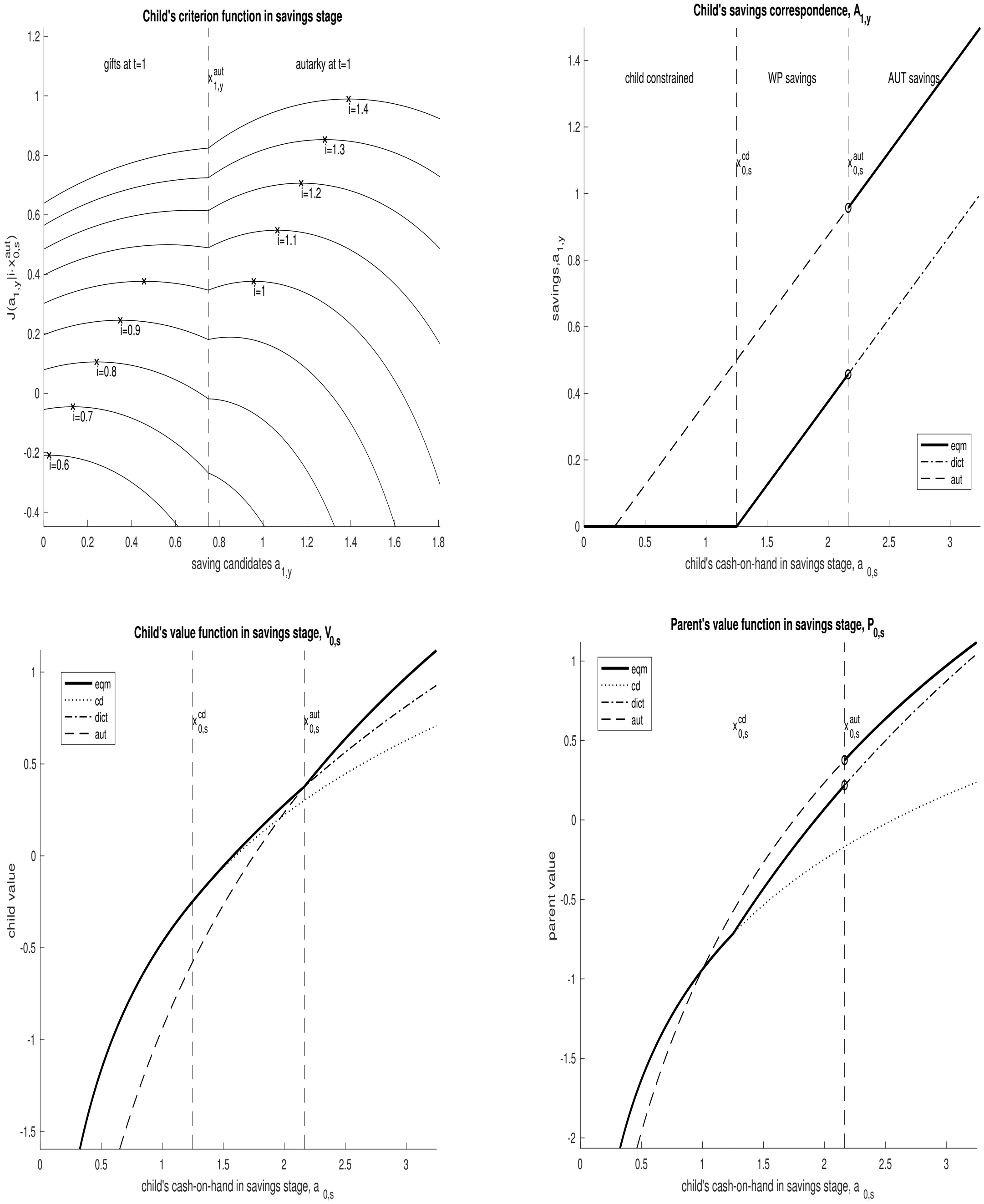
2.2.2 Gift-giving stage

The parent's problem in the initial period's gift-giving stage is given by

$$P_{0,g}(a_{0,g}) = \max_{a_{0,s}\in[a_{0,g},a_{0,g}+y^p_0]} K(a_{0,s}; a_{0,g}),$$

[8] If Cond. 1 does not hold, then the proposition still holds when setting $x^{aut}_{0,s} = -\infty$, since any savings policy by the child leads to autarky at $t = 1$.

Figure 2: $t = 0$: Savings stage



Equilibrium and auxiliary outcomes in savings stage of the initial period. (a) Child's payoff J as a function of next-period savings, $a_{1,y}$, for selected values of current-period cash-on-hand, $a_{0,s} = i \cdot x_{0,s}^{aut}$. (b) Child's savings policy, (d) child's value functions, and (d) parent's value function. Thresholds values are given by $x_{0,s}^{cd} = 1.25$ (child becomes unconstrained) and $x_{0,s}^{aut} = 2.16$ (aut region starts to dominate dict region for child). Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and $y^k = 1/4$.

where the parent's criterion function is given by

$$K(a_{0,s}; a_{0,g}) = u(y_0^p + \underbrace{a_{0,g} - a_{0,s}}_{=-g_0}) + P_{0,s}(a_{0,s}) \tag{14}$$

Just as is the case in the final period, the parent chooses child's next-stage cash-on-hand subject to not being able to extract resources from the child. Again, this decision problem is non-standard, as it was in the child's savings decision; however, now there is not only a kink but also a discontinuity in $K(\cdot; a_{0,g})$, which the criterion inherits from the continuation value $P_{0,s}(\cdot)$. This is apparent in the upper left panel of Fig. 3, which plots $K(\cdot; a_{0,g})$ for fixed levels of $a_{0,g}$.[9] We clearly see the upward jump when the autarky regime is entered and the kink at the threshold where the child switches from being constrained to dictator-savings.

Now, first-order conditions are not even necessary for a global optimum. Before we describe how we find the optimal cash-on-hand policy, it is useful to first establish its monotonicity:

Lemma 5 (Increasing cash-on-hand correspondence at $t = 0$)**.** *Under Ass. 1, the optimal cash-on-hand correspondence $A_{0,s}(a)$ is increasing in the sense of Lemma 4: If A is optimal for a given state a, then the parent will not choose gifts below A for higher states. To be precise, if $A \in A_{0,s}(a)$ for fixed a, then $A - \epsilon \notin A_{0,s}(a + \delta)$ for any $\epsilon > 0$, for all $\delta > 0$.*

A direct corollary of this lemma is that the sequencing of regimes is the same as in the child-savings stage:

Corollary 2.1 (Sequence of regimes in gift-giving stage at $t = 0$)**.** *The sequence of regimes in the first-period gift-giving stage is (i) constrained, (ii) wealth-pooling and (iii) autarky, where (i) or (ii) or both may be skipped. Specifically, there exist numbers $x_{0,g}^{aut} \geq x_{0,g}^{cd} \geq 0$ such that (i) for states $a_{0,g} < x_{0,g}^{cd}$ the equilibrium is such that the child will be constrained, (ii) for states $x_{0,g}^{cd} < a_{0,g} < x_{0,g}^{aut}$ the child chooses positive savings and gifts flow in the final period, and (iii) for states $a_{0,g} > x_{0,g}^{aut}$ the child is in autarky in the final period.*

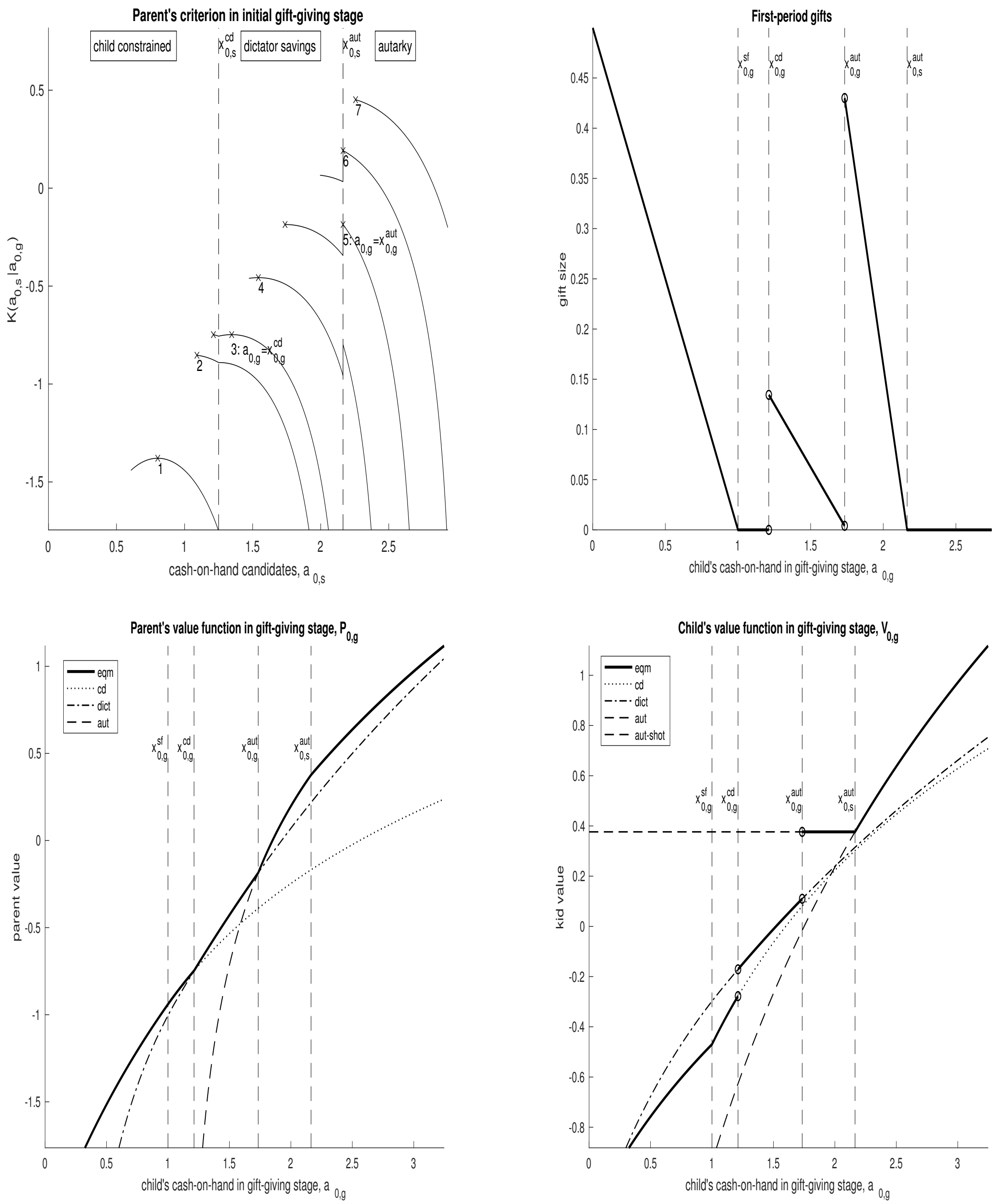
Remark: On the boundaries between two regimes, the parent is indifferent between the adjacent regimes and either policy is compatible with equilibrium.

Our strategy to find the optimal cash-on-hand policy is now the following: (i) Find the local maximum within each regime, which can (usually) be done using first-order conditions.[10] (ii) Find

[9] To generate Fig. 3, we have chosen parameters that lead to a large number of regimes to allow for a comprehensive discussion; in Section 3 we will focus on parameter configuration under which certain regimes disappear, which are interesting in their own right.

[10] This is *always* true for the constrained and autarkic regime. For the dictator-savings region, we can show that under power utility K is differentiable and concave. For other felicity functions $u(\cdot)$, however, K may be non-concave or even discontinuous in the dictator-savings regime if the child's savings policy is convex or discontinuous on this range.

Figure 3: $t = 0$: Gift-giving stage



Equilibrium and auxiliary outcomes in gift-giving stage of the initial period. (a) Parent's payoff K as a function of child's next-stage cash-on-hand $a_{0,s}$ for selected values of current-stage cash-on-hand $a_{0,g}$. (b) Gift-giving correspondence, (c) parent's value function, and (d) kid's value function. Thresholds values are given by $x_{0,g}^{sf} = 1$ (spoon-feeding transfers stop), $x_{0,g}^{cd} = 1.21$ (dict region starts to dominate cd region for parent), $x_{0,g}^{aut} = 1.73$ (aut region starts to dominate dict region for parent) and $x_{0,s}^{aut} = 2.16$ (aut region starts to dominate dict region for child in the savings stage). Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and $y^k = 1/4$.

the thresholds at which the parent is indifferent between the local maxima of neighboring regimes (value-matching), which give us the regime changes $x^{cd}_{0,g}$ and $x^{aut}_{0,g}$ in Corollary 2.1.

For step (i), we need to calculate the derivative of the criterion, $K'(a_{0,s}; a_{0,g}) = -u'(y^p_0 + a_{0,g} - a_{0,s}) + P'_{0,s}(a_{0,s})$, on its smooth parts. Particular care has to be taken when evaluating the parent's marginal continuation. Whenever $A'_{1,y}(a_{0,s})$ exists[11], it equals[12]

$$\begin{aligned} P'_{0,s}(a_{0,s}) &= \alpha u'(c^k_0)(1 - A'_{1,y}(a_{0,s})/R) + \beta P'_{1,y}(A_{1,y}(a_{0,s}))A'_{1,y}(a_{0,s}) \\ &= \alpha u'(c^k_0)(1 - A'_{1,y}(a_{0,s})/R) + \alpha\beta u'(c^k_1)A'_{1,y}(a_{0,s}) \\ &= \alpha u'(c^k_0) - \alpha A'_{1,y}(a_{0,s})\big[u'(c^k_0)/R - \beta u'(c^k_1)\big]. \end{aligned} \tag{15}$$

When going from the first to the second line, we substitute $P'_{1,y}(A_{1,y}) = P'_{1,g}(A_{1,y} + y^k) = \alpha u'(c^k_1)$, which follows from Eq. (10) in Lemma 2; the third line only groups terms. We will now show that we can further simplify this expression to obtain[13]

$$P'_{0,s}(a_{0,s}) = \alpha u'(c^k_0) + \mathbb{I}\{a_{0,s} \in (x^{cd}, x^{aut}_{0,s})\}\beta \underbrace{u'(c^p_1)}_{=\alpha u'(c^k_1)} \underbrace{[1 - A'_{1,s}(a_{1,g})]}_{=c^{p\prime}_1(a^1_g)\in(0,1)} \underbrace{A'_{1,y}(a_{0,s})}_{>0} \tag{16}$$

First, notice that when the child is constrained, then there is no savings response and we have $A'_{1,y}(a_{0,s}) = 0$ in (15); the parent's marginal value from the gift is then fully captured by the child's current marginal utility and $P'_{0,s}(a_{0,s}) = \alpha u'(c^k_0)$. Similarly, if the kid chooses autarkic savings, then the Euler Equation $u'(c^k_0) = R\beta u'(c^k_1)$ must hold, and again the parent's marginal value is $P'_{0,s}(a_{0,s}) = \alpha u'(c^k_0)$. However, when the child engages in dictator-savings, we need to take care of the distortion that future gifts introduce in the child's optimality condition. Using the child's distorted Euler equation (13) and the parent's first-order condition for final-period gifts (4), one obtains the correction term in (16) that is only active in the dictator-savings region.[14] In this

[11] The derivative $A'_{1,y}$ surely exists inside the constrained and autarkic regimes, since $A'_{1,y}(a_{0,s}) = 0$ for $a_{0,s} < x^{cd}_{0,s}$ on the constrained range and since for $a_{0,s} > x^{aut}_{0,s}$ the derivative $A'_{1,y}(a_{0,s}) = A^{aut\prime}_{1,y}(a_{0,s})$ exists by standard arguments. In the dictator-savings regime, i.e. for $a_{0,s} \in (x^{cd}_{0,s}, x^{aut}_{0,s})$, differentiability is not assured in general, but exists for power utility. Finally, at the thresholds $a_{0,s} \in \{x^{cd}_{0,s}, x^{aut}_{0,s}\}$ the derivative does not exist.

[12] Here, $c^k_0 = a_{0,s}(1 - A_{1,y}(a_{0,s}))$ and $c^k_1 = A_{1,s}(A_{1,y}(a_{0,s}))$ are understood to be the child's consumption values that follow as best responses in the game.

[13] Again, $a_{1,g} = A_{1,y}(a_{0,s}) + y^k_1$ and, following up, $c^p_1 = y^p_1 + a_{1,g} - A_{1,g}(a_{1,g})$ are the best responses on the equilibrium path; similar for c^k_0 and c^k_1, see Footnote 12

[14] Specifically, the derivation is

$$\begin{aligned} P'_{0,s}(a_{0,s}) &= \alpha\left(u'(c^k_0) - [\beta u'(c^k_1)A'_{1,s}(a_{1,g}) - \beta u'(c^k_1)]A'_{1,y}(a_{0,s})\right) \\ &= \alpha\left(u'(c^k_0) + \beta u'(c^k_1)[1 - A'_{1,s}(a_{1,g})]A'_{1,y}(a_{0,s})\right) \\ &= \alpha u'(c^k_0) + \beta u'(c^p_1)[1 - A'_{1,s}(a_{1,g})]A'_{1,y}(a_{0,s}). \end{aligned}$$

region, one component of the parent's marginal value is again given by the marginal utility of child's current-period consumption; however, the correction term tells us that we also have to take into account how much more the parent can consume tomorrow (captured by $c_1^{p'}$) due to the fact that the child saves more today (captured by $A'_{1,y}(a_{0,s})$). The correction term is always positive, since it is an additional benefit the parent derives from gift-giving. The correction term helps us to understand the downward kink in the parent's criterion at point $x_{0,s}^{cd}$, where the kid switches from being constrained to dictator savings: The parent suddenly has an additional incentive to give and the slope of K changes when the regime switches.

We now return to Fig. 3 to explain the parent's optimal gift policy.

Constrained regime. Let us first consider very low levels of child cash-on-hand $a_{0,g}$. For these, the optimal gift will be such that the child is constrained and consumes all of the gift, which corresponds to lines 1-2 in the upper left panel. Two cases are possible here: An interior optimum can occur, as is the case for line 1. Gifts are then positive, which corresponds to the region left of $x_{0,g}^{sf}$ in the upper-right panel. This is the typical spoon-feeding gift that the literature (and AHK in particular) has focused on: The child consumes hand-to-mouth and thus the parent effectively controls the child's consumption. At some point of the constrained regime, however, the non-negativity constraint on gifts can bind (as in criterion 2 in the upper left panel) and gifts become zero, as is the case on $[x_{0,g}^{sf}, x_{0,g}^{cd}]$ in the upper-right panel. In the lower two panels, we see that at the switch from positive to zero gifts within the constrained region the parent's value function is smooth (due to the Envelope Theorem), whereas a new downward kink is introduced into the child's value function, as was already the case with final-period gifts.

Dictator-savings regime. At the point $x_{0,g}^{cd}$, the parent is then indifferent between the best option in the constrained and dictator-savings regimes, which is the case for criterion 3 in the upper-left panel. The parent then switches to a positive gift that takes the child into the dictator-savings regime, corresponding to the first spike of the gift function at $x_{0,g}^{cd}$. Since the parent is indifferent, her value $P_{0,g}$ is continuous (but has a kink) at $x_{0,g}^{cd}$. The child's value function, however, has an upward jump when the regime change occurs since the child strictly prefers the higher gift. When increasing $a_{0,g}$ further above $x_{0,g}^{cd}$, the parent decreases gifts. There can be another kink in the gift-giving function within dictator-savings regime if the non-negativity constraint on gifts binds, but this is not the case for the parameters chosen in Fig. 3.[15]

Autarky regime. Finally, at the threshold $x_{0,g}^{aut}$ the parent is indifferent between the best choice on the dictator-savings range and shooting the child to the autarky threshold, corresponding to criterion 5 in the upper left panel and the second spike of the gift function. Since the parent is indifferent at the threshold, the parent's value function is continuous (with a kink) at $x_{0,g}^{aut}$. The

[15] When decreasing curvature to γ to 0.8, for example, this kink is present. We chose log-utility for the figure since the results are easier to interpret.

child, however, strictly prefers the larger gift and there is another upward jump in her value function at this threshold. For child cash-on-hand above $x_{0,g}^{aut}$, the parent's optimal policy is then to shoot the child to autarky, at least as long as this is necessary: Line 6 in the upper left panel depicts another shot to autarky, whereas for line 7 the shot is not necessary and the optimal gift is zero, corresponding to the area right of $x_{0,s}^{aut}$. The child value function is flat on the range $(x_{0,g}^{aut}, x_{0,s}^{aut})$ since all situations lead to the same outcome for the child; then there is a kink at $x_{0,s}^{aut}$ once gifts become zero.

One may now ask how stable a feature the discontinuities in the gift function are. It turns out that the second spike in the gift policy, shots to autarky, are a very robust feature of this environment (since the discontinuity in the parent's continuation value is):

Proposition 2.2 (Shots to autarky)**.** *Under Ass. 1 and Cond. 1, there is a range of initial states, $a_{0,g} \in (x_{0,s}^{aut} - \epsilon, x_{0,s}^{aut})$ with $\epsilon > 0$, for which the equilibrium is such that the parent gives a gift at $t = 0$ and the child is in autarky at $t = 1$.*

Proof. Fix $\epsilon > 0$ small and consider the range of states $a_{0,g} \in (x_{0,s}^{aut} - \epsilon, x_{0,s}^{aut})$. The payoff of shooting the child to autarky can be lower-bounded for all $a_{0,g}$ on this range by $L^{aut}(\epsilon) = P_{0,s}^{+}(x_{0,s}^{aut}) + u(y_0^p - \epsilon)$, since at most ϵ must be given for any $a_{0,s}$ to reach autarky.[16] Similarly, the payoff of any policy that maintains the child in the dictator-savings regime can be upper bounded by $U^{dict}(\epsilon) = P_{0,s}^{-}(x_{0,s}^{aut}) + u(y_0^p)$.[17] As we let $\epsilon \to 0$, we obtain $L^{aut}(\epsilon) > U^{dict}(\epsilon)$ since $u(y_0^p - \epsilon) \to u(y_0^p)$ but the function $P_{0,s}$ has an upward jump, i.e. $P_{0,s}^{+}(x_{0,s}^{aut}) - P_{0,s}^{-}(x_{0,s}^{aut}) = \delta > 0$ by Prop. 2.1. Thus, the parent's optimal policy must be a gift that at least makes the child reach the autarky region. ■

In the example of Fig. 3, all shots to autarky are "point-landings": The parent provides just enough so that the child stays autarkic. One may ask if there can be situations in which the parent chooses interior solutions within the autarky region. The following proposition gives conditions under which this can occur, which are useful in computing results of the model:

Proposition 2.3 (Interiority of gifts-to-autarky)**.** *Suppose Ass. 1 and Cond. 1 hold. Consider gifts leading into autarky in the initial-period gift-giving stage.*

1. ***(point landings)** If $\alpha u'(c_0^{k,aut}(x_{0,s}^{aut})) \leq u'(y_0^p)$, then all such gifts shoot the child exactly to the boundary of the autarky region. In other words, there is always a boundary solution to the parent's problem: $A_{1,s}(a) > a$ and $A_{1,s}(a) \geq a_{1,s}^{aut}$ implies $A_{1,s}(a) = \min\{a_{1,s}^{aut}, a\}$.*

[16] $P_{0,s}^{+}(x)$ denotes the right limit of $P_{0,s}$ at x here. We assume in this proof that the child chooses autarky at $x_{0,s}^{aut}$ when indifferent, which simplifies the exposition. The argument has to be modified slightly if dictator-savings is selected as the equilibrium policy at $x_{0,s}^{aut}$.

[17] Herre, $P_{0,s}^{-}(x)$ denotes the left limit at x.

Table 1: Summary on regimes and smoothness properties by stage

Stage	$t=1$ consumption	gift-giving	$t=0$ savings	gift-giving
max. # regimes	1	2	3	7
FOC necessary	yes	yes	yes	no
FOC sufficient	yes	yes	no	no
max. # policy jumps	0	0	1	2
max. # policy kinks	0	1	1	7
max. # parent-value jumps	0	0	1	0

2. *(**interior solutions**) If $\alpha u'(c_0^{k,aut}(x_{0,s}^{aut})) > u'(y_0^p)$, then (i) some shots to autarky go into the interior of the autarky region and (ii) there exist positive gifts for some starting conditions above the child's autarky threshold. To be precise, there exists an interval $\mathcal{I} = (x_{0,s}^{aut} - \epsilon_1, x_{0,s}^{aut} + \epsilon_2)$, for some $\epsilon_1 > 0$ and $\epsilon_2 > 0$, such that $A_{0,s}(a) > x_{0,s}^{aut}$ and $A_{0,s}(a) > a$ for all $a \in \mathcal{I}$, i.e. gifts are interior solutions.*

*(**Sufficient condition for point landings**) Furthermore, if $u'(y_0^p) \geq \beta R u'(y_1^p)$ – i.e. if the parent would not want to save at the market rate R – then only Case 1 is possible .*

Proliferation of regimes, kinks and jumps. Finally, it is worthwhile to analyze the pattern that has emerged for the multiplication of regimes, kinks and jumps. Table 1 provides an overview. Within any smooth regime in a stage, the non-negativity constraint on gifts or savings can introduce a downward kink (i.e. a strong convexity) in the policy. This leads to a downward kink of the other player's value function. Due to this downward kink, the other player's policy in the preceding period can have an upward jump when the global maximum jumps from one regime to the next. This jump leads to an upward jump in the other player's value function. Regimes that are characterized by value-function jumps at their boundary can even split up in three new regimes, as is the case for the autarkic regime in the initial gift-giving stage, where the parent policy can be a left corner solution (shot-to-autarky), an interior solution (shot into autarky), or a right corner solution (zero gifts). In the next section, we will study different regime configurations that are interesting in their own right from an economic point of view.

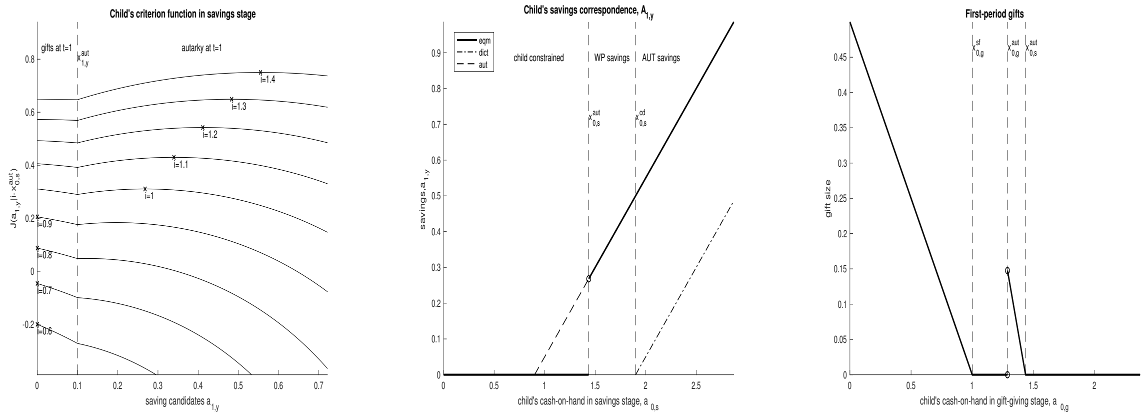
3 Applications

3.1 Consumption floor

There is an interesting parallel between a (one-player) savings model with a consumption floor and the altruism model.[18] The well-known feature of a consumption-floor model is that it distorts

[18]By a consumption floor, we mean a means- and asset-tested subsidy, usually provided by the government.

Figure 4: First period: Consumption floor

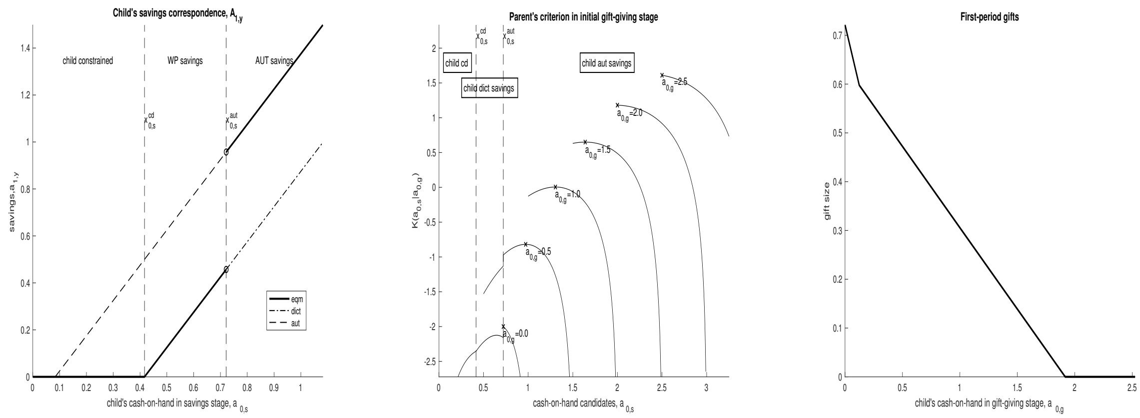


Child goes directly from constrained to autarkic savings. (a) Child's criterion in savings stage, (b) child's savings correspondence, and (c) parent's first-period gift-giving correspondence. Parameterization: logarithmic utility, $y^p = 1, y^k = 0.9, \alpha = \beta = R = 1$.

savings decisions as in certain states of the world the marginal benefit of savings is zero. For example, if the consumption floor is such that resources are filled up any time resources fall below a certain threshold level, then any savings that lead to resources which are below this floor are wasted. As a result consumption today is increased in order to deplete all resources. This over-consumption shares important similarities with the over-consumption which takes place in the dictator-savings region when the child saves anticipating future transfers. The child knows that an additional unit of cash-on-hand will not translate into an additional unit of resources tomorrow since the parent will trim the transfer amount by some fraction. Hence, in the altruism model trimming is partial and not complete, but the incentive effect is similar.

We will now show that the altruism model can generate an equilibrium that looks very similar to the consumption floor in its dynamics. To do this, we increase the child's income, y^k, with respect to our baseline case. Figure 4 shows that under this parameterization, the child switches from being constrained directly to the autarky regime, i.e. the dictator-savings regime is skipped. The child knows that it will only receive a small gift, precisely 0.05, in the final period when not saving, which will be crowded out when the child starts saving. Thus the child optimally jumps directly to autarky. Finally, the initial-period gift-giving policy in Fig. 4 shows that spoon-feeding gifts (that are similar to the consumption floor) occupy a large region of the state space, while shots to autarky are of course still present in accord with Prop. 2.2. Besides this being an interesting analogy, this also tells us that some of the complications present in the altruism model are already present in simpler models.

Figure 5: First period: College case



Child always engages in autarkic savings. (a) Child's savings correspondence, (b) parent's criterion in gift-giving stage, and (c) parent's first-period gift-giving correspondence. Parameterization: logarithmic utility, $y^p = 1, y^k = 0.25, \alpha = \beta = 1, R = 3$.

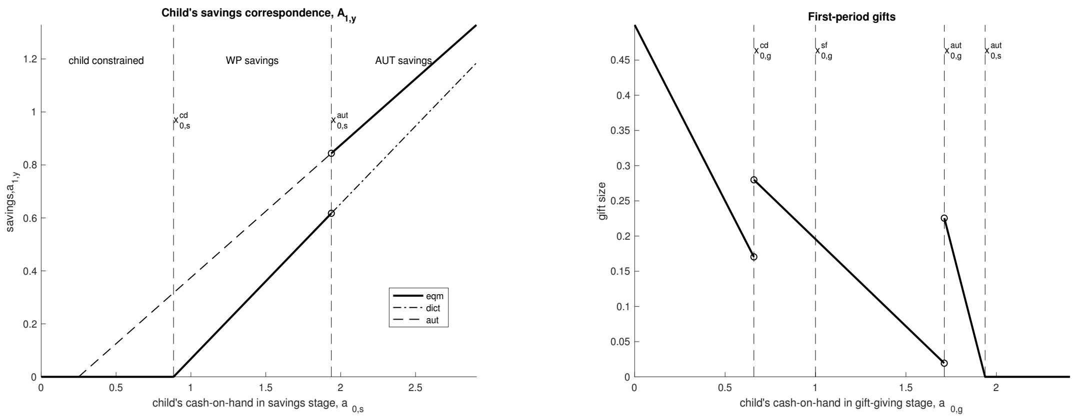
3.2 College case

Next, we will now consider outcomes of the game in which i) gifts flow only in the initial period, i.e. the game always ends in the autarky regime and ii) the parent provides initial gifts that go *into* the autarky region and not just to its threshold, i.e. the parent effectively saves through (or invests in) the child. We will argue that parents' investments in child education could be a candidate for this type of equilibrium.

In order to find parameters that lead to i) and ii), we note that the sufficient condition in Prop. 2.3 tells us that a high interest rate R is likely to lead to such an outcome. Figure 5 shows the case when $R = 3$ and the remaining parameterization is as before. When child's cash-on-hand is low, the parent catapults the child exactly to the cut-off value where the child just engages in autarkic savings, $x_{0,s}^{aut}$. For higher levels of cash-on-hand, transfers bring the child within the autarkic savings region. Finally, once the non-negativity constraint on gifts starts to bind, we enter the third sub-regime within the autarkic regime in which gifts are zero.

For the savings-through-the-child equilibrium to arise one assumption is crucial: The parent does not have access to the savings technology with the high return R. If she had, the parent would use own savings, which is a more effective way of increasing the parent's future consumption than through child savings. Thus, an empirical application of this type of equilibrium has to be for an asset that i) has a high return and ii) only the child has access to. The child's college education may be just this.

Figure 6: First period: $\gamma = 2$



Child's savings correspondence and parent's first-period transfer correspondence with power utility where the coefficient of relative risk aversion equals $\gamma = 2$. The thresholds in the savings stage are: $x^{cd}_{0,s} = 0.88$ (child becomes unconstrained), $x^{aut}_{0,s} = 1.94$ (aut savings region starts to dominate dict savings region for child). The thresholds in the gift-giving stage are: $x^{sf}_{0,g} = 1$ (spoon-feeding transfers stop), $x^{cd}_{0,g} = 0.66$ (dict region starts to dominate cd region for parent), $x^{aut}_{0,g} = 1.71$ (aut savings region starts to dominate dict savings region for parent). Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and $y^k = 1/4$.

3.3 Changing the inter-temporal elasticity of substitution

In all our previous examples we have made use of logarithmic utility. We will now study how results change when changing the curvature of the felicity functional, $u(c) = c^{1-\gamma}/1 - \gamma$. Appendix B shows the solution. Optimal policies within regimes are linear and can be derived in closed form; the cut-offs between regimes can be derived either in explicit or implicit form. It turns out that curvature matters only in the initial (but not the final) period, and it does so through its effect on the inter-temporal elasticity of substitution.[19]

Figure 6 shows the policies in the initial period when the coefficient of relative risk aversion equals 2, a commonly used value in the macroeconomics literature. One notable change is that the slopes of the optimal policies have changed, a feature we will explain now.

It is instructive to consider the closed-form expressions for the child's marginal propensity to

[19] Specifically, in the final stage a sufficient statistic to calculate equilibrium policies is the fraction of family wealth that the parent wants to assign to the child, which we define as $\hat{\alpha} \equiv \alpha^{1/\gamma}$. Since the parent chooses child's cash-on-hand such that this ratio is implemented, any α-γ combination leading to the same $\hat{\alpha}$ leads to identical final-stage gifts.

consume within the dictator- and the autarkic-savings region,

$$MPC^{dict} = \frac{1}{1 + \beta^{1/\gamma} \tilde{R}^{(1-\gamma)/\gamma}}, \tag{17}$$

$$MPC^{aut} = \frac{1}{1 + \beta^{1/\gamma} R^{(1-\gamma)/\gamma}}, \tag{18}$$

$$\text{where } \tilde{R} = \frac{\alpha^{1/\gamma}}{1 + \alpha^{1/\gamma}} R. \tag{19}$$

A key insight when deriving these expressions is that the dictator-savings regime can be construed as a standard savings problem but with the twist that the interest rate $\tilde{R}$ is lowered with respect to the autarkic problem, R, since the altruistic parent "taxes" savings.[20] Notice that $\tilde{R}$ is larger the more altruistic the parent is: a less-altruistic parent "taxes" savings more, since she assigns a lower fraction of (marginal) family wealth to the child. The above expressions directly lead us to the following result:

Proposition 3.1 (Child's marginal propensity to consume)**.** *Under Ass. 2 (power utility), the marginal propensities to consume in the autarkic and dictator regimes of the savings stage, defined in Eq.* (18) *and* (17)*, satisfy the following:*

1. *If $\gamma < 1$ (high intertemporal elasticity of substitution), then $MPC^{aut} < MPC^{dict}$ and MPC^{dict} is decreasing in α, i.e. the marginal propensity to consume decreases as the parent becomes more altruistic.*

2. *If $\gamma = 1$ (unit elasticity), then $MPC^{aut} = MPC^{dict}$ and $MPC^{dict} = 1/(1+\beta)$ is independent of the degree of altruism, α.*

3. *If $\gamma > 1$ (low intertemporal elasticity), then $MPC^{aut} > MPC^{dict}$ and MPC^{dict} is increasing in α, i.e. the marginal propensity to consume* ***increases*** *as the parent becomes more altruistic.*

Proof. First, note from Eq. (19) that i) $\tilde{R} < R$ for any $\alpha > 0$ and $\gamma > 0$, and ii) $\tilde{R}$ is increasing in α for any $\gamma > 0$. The comparative-statics results in the proposition then follow directly from Eq. (18) and (17).[21] ■

We first note that the marginal propensities to consume are equal across regimes and invariant to altruism *only* in the case of logarithmic utility, $\gamma = 1$, as is the case in our previous numerical example, see Figure 2. Savings rates are constant which is akin to the familiar result for log-utility that the income and the substitution effect cancel out.

[20] Also the endowment is altered in this alternative problem.

[21] For the derivation of the MPCs under power utility, see Appendix B.

A striking implication arises for the empirically relevant case $\gamma > 1$. Prop. 3.1 says that the child is more prone to consume a marginal unit of wealth in autarky than when expecting gifts. Put differently, the child's marginal propensity to *save* is lower in autarky than in the dictator regime. At first sight, this appears to contradict the over-consumption result known as the Samaritan's dilemma, i.e. the fact that savings in the dictator regime are discouraged by the altruistic "tax". However, the MPC is about marginal *changes* in savings and not about the *level* of savings, which, of course, increases as the jump in the savings policy in the figures show. To understand the differing MPCs, it is instructive to consider the limiting case $\gamma \to \infty$, in which case the preferences converge to Leontief. Now, the child implements a fixed ratio between first- and second-period consumption, e.g., consumption is equalized when $\beta = 1$. If the interest rate is low, the child has to increase its savings by more in the first period in order to maintain the fixed consumption ratio which explains why the marginal propensity to save is larger in the dictator than in the autarkic regime.

Other notable changes apparent in Fig. 6 is that the thresholds $x_{0,s}^{cd}$ and $x_{0,s}^{aut}$ have both moved left. This is because with a lower IES the child is less inclined to substitute high current consumption for relatively low future consumption.

Furthermore, the slopes in the parent's initial gift policy are now also affected. As the expressions (given in the appendix) are more convoluted it suffices to say that the model is able to generate a wide variety of marginal propensities to give when varying γ. As for the regime changes in the parent's gift-giving policy, we note that $x_{0,g}^{cd}$ has shifted left with respect to the log-utility case, i.e. the parent lifts the child from constrained into the dictator-savings region already at lower cash-on-hand levels. This is mainly due to the child's marginal propensity to save being higher in the dictator regime, which makes the convexity (downward kink) in the parent's value function more pronounced and increases the parent's incentives to "save through the child", which is captured by the higher $A'_{1,y}$ in Eq. (16).

4 Extending the baseline model

In this section we show through a series of extensions that the key results obtained from our baseline model are robust.

4.1 Parent can save

The reason we left out the parent's savings decision in the baseline model is not for realism but because we obtain a more tractable framework (reducing the dimensionality of the state space by one) without losing anything of qualitative significance. However, we will now show that the same strategic considerations remain intact when the parent can save.

Thus, for this section consider the following modification to the baseline setting. The sequence of decisions is now as follows. In the initial period, the parent first chooses gifts and savings (at the same interest rate R and subject to a no-borrowing constraint), and, second, the child chooses savings. The final period is identical to before. The child's payoff-relevant state in the child-savings stage is given by $(a^k_{0,s}, a^p_{1,y})$, where $a^k_{0,s} = a^k_{0,g} + g_0$ is as before and $a^p_{1,y}$ is the parent's savings choice made in the first stage. First, it turns out that we can recycle one of our results from before for the child-savings stage, since we can treat the parent's (fixed) savings as a part of her final-period endowment:

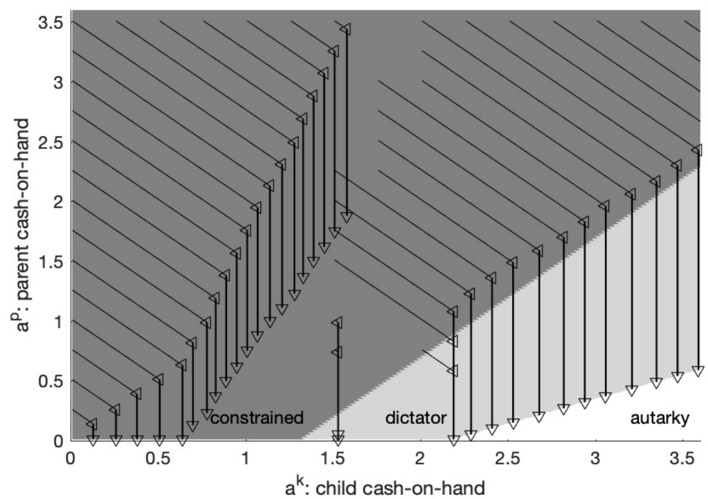
Corollary 4.1 (Discontinuities when parent can save)**.** *Consider an alternative environment in which the parent can also allot resources to savings (besides gifts and consumption). In the initial period's child-savings stage, the policy function $A_{1,y}(\cdot)$ and the parent value function $P_{0,s}(\cdot)$ display jump continuities. Specifically, for any fixed parent cash-on-hand a^p there exists an upward jump in $A_{1,y}(\cdot, a^p)$ and $P_{0,s}(\cdot, a^p)$ at some level of child cash-on-hand a^k.*

Proof. In the alternative environment, let $a^p_{1,g}$ denote the parent's cash-on-hand in the beginning of the final period's gift-giving stage, which consist of the parent's savings plus its endowment. We have to extend the state vector and add parent cash-on-hand $a^p_{1,g}$ in period 1's gift-giving stage and period 0's child-savings stage. But now, note that when fixing the parent's savings choice $a^p_{1,g}$, we can set $y^p_1 = a^p_{1,g}$ in the original environment (without parent savings) and apply Prop. 2.1 to show the desired results. ■

This result tells us that exactly the same kind of discontinuity as in the baseline setting obtains for the child's savings policy and the parent value function entering this stage. It is thus unsurprising that when going back to the initial gift-giving plus parent-savings stage, again similar dynamics play out. Still, in this stage matters become somewhat more complicated and we cannot directly use our results from before. Computationally, however, we find the same kinds of discontinuities as in the baseline setting. We will illustrate them now in an example that we solve numerically by brute-force maximization on a discrete grid, which ensures that the algorithm can deal with the discontinuities in the value functions present.

Figure 7 shows the parent's gift and savings policy in the initial stage. First, positive first-period gifts are shown as movements in the state space along the diagonal vectors. Any $(a^k_{0,g}, a^p_{0,g})$ that falls on a diagonal line is moved to the tip of the vector in the gift-giving stage; this is a common property of altruism models. If one gives a gift that takes the economy to a state in which one would have chosen to give gifts, too, then one should surely extend the initial gift just as long as transferring wealth is beneficial. Second, we show the movement in the state space due to parent savings, conditional on the parent having provided a positive gift (to avoid cluttering the diagram),

Figure 7: $t = 0$: Parent can save



Dynamics induced by parent's decision in initial period when parent can save. Diagonal vectors show displacements in state space due to first-period gifts and vertical vectors show movements in state space due to parent's savings choices. Shaded areas correspond to regimes in ensuing child-savings stage, i.e. they mark the optimal savings choices of the child in the savings stage after first-period transfers and parental savings have taken place. Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and $y^k = 1/4$.

by the vertical vectors. Finally, the tip of the vertical vector represents the payoff-relevant state for the child in the ensuing child-savings-stage. The shading of the area then indicates the regime in the savings stage, i.e. whether the child is constrained, chooses dictator savings or autarkic savings.

Overall, the figure reflects the fact that the main features of the initial period uncovered in the baseline model remain present when the parent can also save. In the left upper corner, the parent is rich relative to the child. First-period spoon-feeding gifts flow that are entirely consumed by the child (since the child remains constrained) and the child expects to obtain final-period gifts. In the right upper corner, shots-to-autarky occur which bring the child exactly to the boundary where it just chooses autarkic savings and so does not receive any final-period gifts. Finally, there are transfers which lift the child out of the constrained region into the dictator-savings region, which show up in the middle on the bottom. As for value functions when entering the game, we find again the same smoothness properties, i.e. various kinks and discontinuities.

4.2 Adding uncertainty

We now return to the baseline setting but include uncertainty over the child's endowment.At this point we need to fully rely on numerical methods. Again, the point is that the key features of the deterministic setting remain intact, even for unrealistically large levels of noise.

Discrete support A common way of modeling income uncertainty in quantitative work is to assume that income shocks follow a discrete-state Markov process. We now provide a simple example to show that a discrete-state process can make matters actually more complicated than in the deterministic setting, i.e. new types of regions can occur and the number of kinks and discontinuities actually increases.

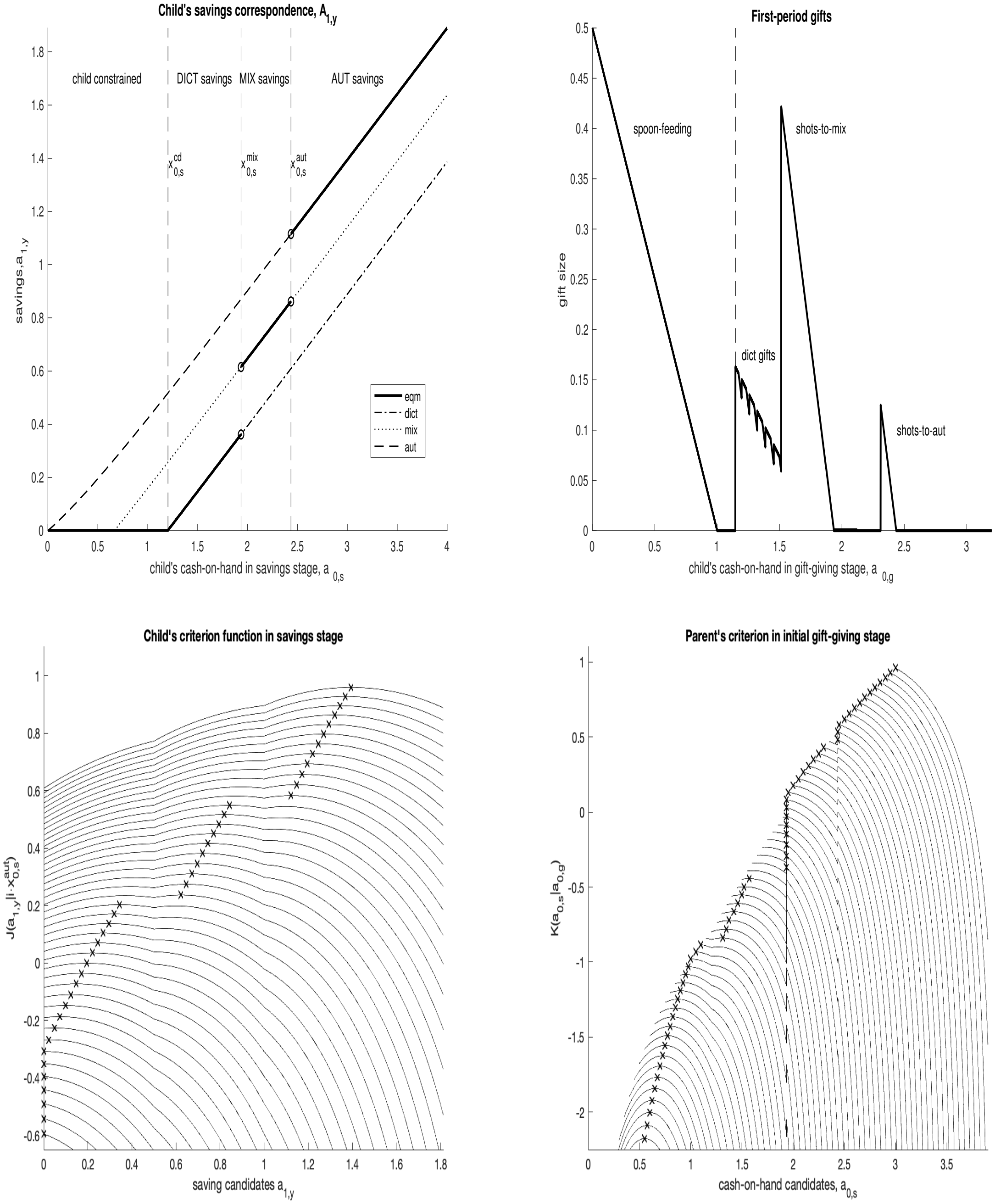
Suppose the same environment as in the baseline model, but assume that child's income in the income stage of the final period is uncertain and realized before the parent makes the final-period gift decision. For simplicity, we will assume that there are only two income realizations, high and low (but the logic carries over to any number of finite states).

Fig. 8 shows the kid's savings correspondence and criterion function on the left-hand side. We observe that as in the deterministic case, the child's payoff function is non-concave which leads to discontinuities in the child's savings policy and the parent's value function. Moreover, matters become even more complicated as there is now an additional region which we refer to as *mixed* savings region. This region exists since for certain levels of savings, the child knows she will receive transfers in the low but not the high income realization in the final period. The remaining savings regions are familiar from the deterministic case: the child can be constrained or engage in dictator savings, in both of these cases it receives final-period transfers with certainty. Last, the child may engage autarkic savings and receive no transfer in the final period. By the logic of the baseline model, the effective return to savings increases discretely as the child moves from the dictator- to the mixed- and then to the autarky-savings region, since the tax on gifts is removed in a state of the world. Hence, there are two downward kinks in the child's criterion (shown in the lower-left panel), leading to *two* discontinuities in the child's savings policy.

Finally, the right-hand side of Fig. 8 shows the parent's first-period gift correspondence and criterion function in the gift-giving stage. The parent's payoff function now has two discontinuities, one at each point where child savings jump up. Furthermore, there is the familiar kink at the point where the child starts to save. This leads to as many as three jump discontinuities in the parent's gift-giving policy (shown in the upper-right panel). Initially, i.e. for low levels of $a_{0,g}$, the parent provides spoon-feeding transfers, followed by zero gifts while the child is constrained. The first upward jump, *dict gifts*, are transfers which send the child from being constrained into the dictator-savings region or occur within the dictator-savings region. The second upward discontinuity is just like the shot-to-autarky in the deterministic case, except that in the example the parent shoots the kid into the region where it engages in mixed savings and is autarkic in *one* of the two states of the world. Shooting the child into autarky is too costly for the parent as the child will choose autarkic savings at only a relatively high level of cash-on-hand. The final upward jump is then the shot-to-autarky.

Continuous support In AHK, the shock follows a continuous distribution; this gives us maximal hope that non-convexities are smoothed. But, we now demonstrate that all of the features uncovered in the deterministic case remain intact, even when choosing very large levels of noise.

Figure 8: $t = 0$: two-state support



Left panel: Child's savings policy and criterion function in savings stage. Right panel: Parent's first-period gift policy and criterion function in gift-giving stage. Child's income in the income stage of the final period is $y_h^k = 0.5$ or $y_l^k = 0$ with probability $pr(y_h^k) = 0.5$. Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and grid size $N = 5,000$.

Suppose now that the child's income in the income stage of the final period follows a log-normal distribution, $\ln y^k \sim N(\mu, \sigma^2)$. Figure 9 shows the situation when the expected value of child's income equals its baseline value (one-fourth) and its standard deviation equals one[22], an empirically unrealistically high level of income uncertainty. We see that the implications of the deterministic model carry over: The child's payoff functions in the savings stage remain non-concave, which is due to the strong convexity in the parent's gift-giving function in the final period. Thus, the same type of discontinuity is present in the child's savings policy, which is when it switches from a local optimum at which she likely receives gifts to another local optimum where she likely stays autarkic. The jump in savings then generates the same discontinuities in the initial gift-giving stage as in the baseline model (shown in the right panels now). Finally, we find that increasing the standard deviation of the child's endowments further is insufficient to smooth out the payoff functions; we do not present these results as they are nearly indistinguishable from the case presented here.[23]

5 Conclusions

In this paper, we have provided a full theoretical characterization of the basic two-period altruism model. Our results carry important consequences.

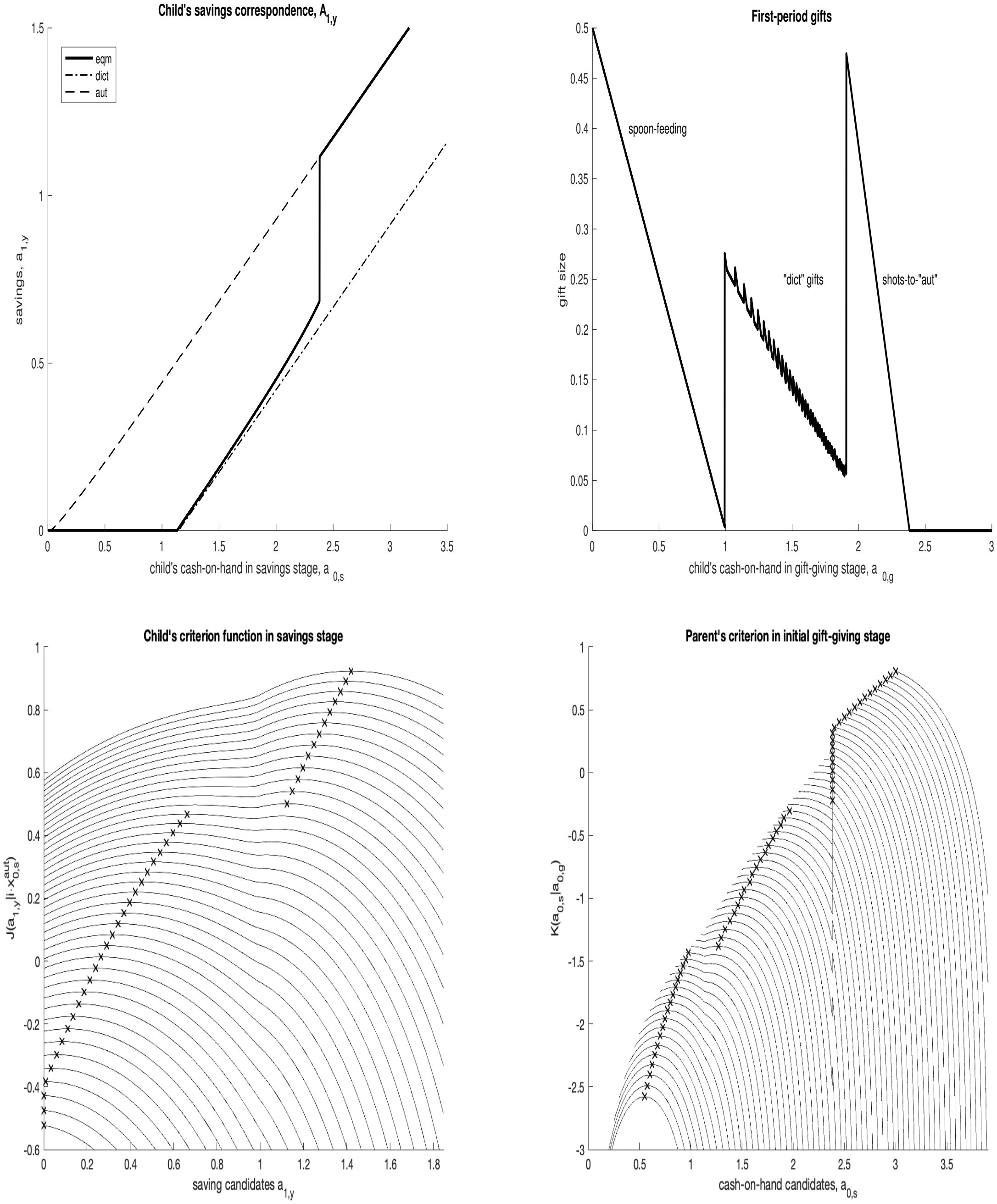
First, the way we think about and characterize operative transfer motives needs to be expanded beyond equalization of marginal utilities. Second, the statement that according to the altruistic hypothesis, richer children should receive smaller transfers, ceteris paribus, needs to be revised. Shots-to-autarky are a robust feature of equilibrium, meaning that it is entirely consistent with altruism that richer children can receive higher transfers. This prediction has been typically ascribed to the exchange-motivation hypothesis of transfers, the argument being that transfers to richer children must be larger in order to compensate them appropriately for services they provide. The argument also extends to how to empirically test the altruistic hypothesis. Third, the presence of uncertainty is unlikely to remove discontinuities in discrete-time altruism models; in fact, uncertainty in the form of discrete shocks can make matters even more complicated. Fourth, the exclusion of parental savings does not fundamentally alter the nature of strategic interactions.

Finally, our results provide two inputs into a recent quantitative-macroeconomics literature that has used discrete-time altruism models with savings. Our results can guide the quest for

[22] I.e. we set $\sigma = 1.68$ such that $std(y^k) = 1$.

[23] Figures 10 to 12 in the Appendix show that when increasing the child's expected endowment, the savings correspondence can become continuous and the child's criterion function becomes concave. The reason is as in the deterministic case when Condition 1 is violated: Transfers in the final period become increasingly less likely. However, even with a continuous savings function the parent's first-period criterion can be convex, since the child's savings function is, and the parent's gift policy can have a jump discontinuity, see Fig. 12.

Figure 9: $t = 0$: continuous support



Left panel: Child's savings policy and criterion function in savings stage. Right panel: Parent's first-period gift policy and criterion function in gift-giving stage. Child's income in the income stage of the final period follows log-normal distribution with expected value equal to 0.25 and standard deviation equal to 1. Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and grid size $N = 5,000$.

appropriate solution algorithms for these models; our results tells us that algorithms should be able to deal with locally convex and even discontinuous value functions.[24] Also, our (near-)closed form solutions give a much-needed benchmark to test such algorithms and to judge their accuracy.

We leave it to future research to i) identify the quantitative importance of the different types of transfers predicted by the model and ii) to find potential applications for the novel types of equilibria we identify, such as the college case in Section 3.2.

[24]Note that if continuous-support shocks are introduced after each stage of the game, value functions will usually be smoothed, but will likely fail to be globally concave.

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A Proofs

Proof. (Lemma 2) Applying the Implicit Function Theorem to Eq. (4) and using it in Eq. (6), we find

$$A'_{1,s}(a_{1,g}) = \frac{u''(c_1^p)}{u''(c_1^p) + \alpha u''(c_1^k)} \in (0,1) \qquad \text{for } a_{1,g} < x_{1,g}^{aut},$$
$$\text{where } c_1^p = y_1^p + a_{1,g} - A_{1,g}(a_{1,g}) \qquad \text{and } c_1^k = A_{1,s}(a_{1,g}).$$

This derivative is continuous since $u''(\cdot)$ is a continuous function by Ass. 1. The slope being 1 for $a_{1,g} > x_{1,g}^{aut}$ follows from Eq. (5). The statement about the downward kink at $x_{1,g}^{aut}$, i.e. Eq. (9), also follows directly from $A_{1,s}(\cdot)$ being equal to $A_{1,s}^{dict}(\cdot)$ below $x_{1,g}^{aut}$ and equal to $A_{1,s}^{aut}(\cdot)$ above. The child's value coming into the stage is then given by $V_{1,g}(a_{1,g}) = u(A_{1,s}(a_{1,g}))$, which inherits the differentiability properties of $A_{1,s}(\cdot)$ by the Chain Rule and since $u(\cdot)$ is continuously differentiable by Ass. 1. As for the statement on the parent's value function, notice that $P_{1,g}^{dict}(\cdot)$ is an upper envelope to $P_{1,g}^{aut}(\cdot)$, since zero gifts is always a feasible option:

$$P_{1,g}^{aut}(a_1) = u(y_1^p) + \alpha u(a_{1,g}) \leq P_{1,g}^{dict}(a_{1,g}) = u(y^p + a_{1,g} - A_{1,s}^{dict}(a_{1,g})) + \alpha u\big(A_{1,s}^{dict}(a_{1,g}))\big),$$

with equality only for $a_{1,g} = x_{1,g}^{aut}$. By the Envelope Theorem, we have $P_{1,g}^{aut'}(x_{1,g}^{aut}) = P_{1,g}^{dict'}(x_{1,g}^{aut})$. Thus the left and right derivative of $P_{1,g}(\cdot)$ at $x_{1,g}^{aut}$ coincide and $V_{1,g}(\cdot)$ is differentiable at $x_{1,g}^{aut}$. Since the child's consumption is increasing in $a_{1,g}$, $P'_{1,g}(a_{1,g})$ is monotone decreasing and thus $P_{1,g}(\cdot)$ is globally concave. Finally, concavity of $V_{1,g}(a_{1,g})$ on the range $(x_{1,g}^{aut}, \infty)$ follows from concavity of $u(\cdot)$ and the fact that the child consumes $a_{1,g}$ for $a_{1,g} > x_{1,g}^{aut}$. ■

Remark: Depending on the shape of the gift function, the child's value function may not be concave in the dictator region. However, once we assume homothetic preferences, the marginal propensity to give is constant and also this function is well-behaved. From the kid's value function $V_{1,g}(a) = u(A_{1,g}(a))$, we find that a sufficient condition for concavity on the wealth-pooling range is

$$\frac{A''_{1,s}(a)}{A'_{1,s}(a)} \leq -\frac{u''(A_{1,s}(a))}{u'(A_{1,s}(a))} \qquad \text{for all } a \in [0, a_{1,g}^{aut}], \tag{20}$$

i.e. the slope of the gift function "should not be too convex", specifically the amount allotted to the child should grow at a rate below the growth rate of marginal utility at this point.

Proof. (Lemma 4) Since savings A are optimal given state a, saving ϵ less must do weakly worse, i.e. we have

$$0 \geq J(A-\epsilon; a) - J(A; a) \qquad \text{for all } \epsilon \in (0, A].$$

Now, writing out the terms of $J(\cdot)$ and using the fact that the marginal cost of savings drops as one becomes richer, we have for any $\epsilon \in (0, A]$:

$$
\begin{aligned}
0 \geq & u\big(a - A/R + \epsilon/R\big) - u(a - A/R) + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
= & \int_{a-A/R}^{a-A/R+\epsilon/R} u'(c)dc + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
> & \int_{a-A/R}^{a-A/R+\epsilon/R} u'(c+\delta)dc + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
= & \int_{a+\delta-A/R}^{a+\delta-A/R+\epsilon/R} u'(c)dc + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
= & u\big(a + \delta - A/R + \epsilon)/R\big) - u(a + \delta - A/R) + \beta V_{1,y}(A - \epsilon) - \beta V_{1,y}(A) \\
= & J(A - \epsilon; a + \delta) - J(A; a + \delta).
\end{aligned}
$$

for any $\delta > 0$. Note here that the strict inequality is justified since $u'(\cdot)$ is a strictly decreasing function by Ass. 1 and since $\delta > 0$. From the above it follows that $J(A - \epsilon; a + \delta) < J(A; a + \delta)$, i.e. $A - \epsilon$ is not optimal for state $a + \delta$. Since $\epsilon \in (0, A]$ and $\delta > 0$ were arbitrary, this establishes the desired result. ■

Proof. (Proposition 2.1) We first prove that as $a_{0,s} \to \infty$, the child will want to save enough to enter the autarky region in order to smooth consumption. Note that the derivative of the child's continuation value is

$$V'_{1,y}(a) = V'_{1,g}(a + y_1^k) = u'(c_1^k)A'_{1,g}(a + y_1^k).$$

The idea now will be to show that the marginal benefit of savings is fixed, while the marginal cost of savings approaches zero as the child gets richer. On the range where gifts are positive, we can bound $u'(c_1^k) \in [u'(x_{1,g}^{aut}), u'(A_{1,g}(0))]$ since the parent's gift policy $A_{1,g}(\cdot)$ is increasing by Lemma 2. By the same lemma, $A'_{1,g}(\cdot)$ is a continuous function on the interval $[0, x_{1,g}^{aut}]$ which satisfies $A'_{1,g}(a) \in (0, 1)$ for all $a \in [0, x_{1,g}^{aut}]$, thus there must exist bounds $0 < \min A'_{1,g} \leq \max A'_{1,g} < 1$ on this derivative by the Weierstrass Theorem. Thus the marginal continuation value $\beta V'_{1,y}(\cdot)$ is lower-bounded by $\beta u'(x_{1,g}^{aut}) \min A'_{1,g}$ in the dictator region, i.e. for $a \in [0, x_{1,y}^{aut}]$. However, as we let $a_{0,s} \to \infty$, the marginal cost of savings $u'(a_{0,s} - a'/R)$ approaches zero by the Inada condition in Ass. 1 for any fixed a' that leads into the transfer region. Hence we will have $J'(a'; a_{0,s}) > 0$ for all $a' \in [0, x_{1,y}^{aut}]$ for $a_{0,s}$ large enough, i.e. the child's $J(\cdot; a_{0,s})$ will be increasing throughout the dictator regime . The optimal policy must thus feature autarky at $t = 1$ for $a_{0,s}$ large enough.

Now, denote the optimal savings policy in the autarky range by

$$\tilde{A}^{aut}_{1,y}(a_{0,s}) = \arg\max_{a' \geq x^{aut}_{1,y}} J(a'; a_{0,s})$$

for all $a_{0,s} \geq x^{aut}_{1,y}/R$, i.e. for all states for which saving into autarky is feasible. Since J is continuous and strictly concave on the autarky range, the maximum is attained by a unique maximizer for each state $a_{0,s}$, thus $\tilde{A}^{aut}_{1,y}(\cdot)$ is a singleton-valued correspondence and thus a function. By Berge's Maximum Theorem, $\tilde{A}^{aut}_{1,y}$ is also continuous. Now, denote the lowest state at which autarky is among the child's optimal policies by

$$x^{aut}_{0,s} \equiv \inf\{a_{0,s} : A^{aut}_{1,y}(a_{0,s}) \in A_{1,y}(a_{y,s})\}.$$

It then follows from increasingness of the savings policy (Lemma 4) that any optimal policy must feature autarky for any state $a_{0,s} > x^{aut}_{0,s}$. As shown above, the savings policy $A_{1,y}(\cdot)$ must thus be a continuous function on the range $a_{0,s} > x^{aut}_{0,s}$. Again by Berge's Maximum Theorem, $\tilde{A}^{aut}_{1,y}(x^{aut}_{0,s})$ must also be optimal at the threshold $x^{aut}_{0,s}$, since upper-hemi-continuity of the optimal policy translates into continuity for a function (or singleton-valued correspondence). Furthermore, upper-hemi-continuity and non-emptiness of the policy correspondence (which are again guaranteed by the Maximum Theorem) on the range $a_{0,s} < x^{aut}_{0,s}$ imply that there must also be a second maximizer $a' \in A_{1,y}(x^{aut}_{0,s})$ with $a' < x^{aut}_{1,y}$ that leads into the dictator regime at the threshold. The value function $V_{0,s}(\cdot)$ is continuous at the threshold $x^{aut}_{0,s}$ and the child is indifferent between the (best) autarkic and the (best) wealth-pooling savings policy, again by the Maximum Theorem.

Also, note that for low enough child cash-on-hand, autarky is not an option since at some point it is not feasible to save into this area. Formally, when $a_{0,s} \to 0$, autarky is not feasible if Condition 1 holds. We also must have that zero savings are optimal for $a_{0,s}$ low enough. But it is not clear if we must always have a region with positive dictator savings – this region might be skipped.

By the Maximum Theorem, the child's value function $V_{0,s}(\cdot)$ is continuous at $x^{aut}_{0,s}$; we will now show that the parent's value function is discontinuous at this point, however. Denote by a^{dict} the maximal amount that the child saves in the wealth-pooling region, i.e. set $a^{dict} = \max\{A_{1,y}(a) : a \in [0, x^{aut}_{0,s}]\}$ Note that the maximum is attained by the Maximum Theorem; Lemma 4 tells us that a^{dict} must be an optimal savings policy at the autarky threshold, i.e. $a^{dict} \in A_{1,y}(x^{aut}_{0,s}$. Also, it must be that a^{dict} takes the economy within the wealth-pooling region and not on the kink, i.e. $a^{dict} < x^{aut}_{1,y}$, since the criterion $J(\cdot)$ has a downward kink at the threshold. But this implies that the parent will give a positive gift and consumption will be strictly lower than under autarky; this argument also implies to all other policies that lead to wealth-pooling, since they imply even lower consumption for the parent by construction of a^{dict}. Now, since the parent's value function equals $P_{0,s}(a) = \alpha V_{0,s}(a) + \beta u(c_1^p)$, the parent's value function has an upward jump when the regime switches to autarky, i.e. we have

$$\sup_{a<x_{0,s}^{aut}} P_{0,s}(a) < \inf_{a>x_{0,s}^{aut}} P_{0,s}(a). \tag{21}$$

Note that we don't make a statement about what occurs at the threshold itself. Since both wealth-pooling and autarky are optimal for the child, either of the two regimes can be played in an equilibrium Also, note that the parent's value function (but not the child's value function) may have further discontinuities within the dictator region since the child may switch from one local maximum to another. Such jumps in savings must always be upward, and jumps in the parent's value must also be upward by the same argument as above. ■

Proof. (Lemma 5) The proof follows exactly the same strategy as the proof of Lemma 4, but taking care of the non-negativity constraints for gifts.

Fix some child cash-on-hand $a \geq 0$ coming into the gift-giving stage at $t = 0$. First, note that if giving no gift is optimal, i.e. $A = a \in A_{0,s}(a)$, then the statement follows trivially since a is not feasible for any state $a + \delta$, for $\delta > 0$, since gifts cannot be negative.

So assume from now on that $A > a$, i.e. the gift is positive. Again, note that the statement in the lemma follows trivially for any δ large enough such that A is not feasible any more, i.e. $a + \delta > A$.

Thus restrict attention to δ small enough such that $a + \delta \leq A$, i.e. setting A is feasible at state $a + \delta$. We now follow the proof strategy from Lemma 4. Since A is optimal at a for the parent, setting ϵ less must do weakly worse, i.e. we have

$$0 \geq K(A - \epsilon; a) - K(A; a) \qquad \text{for all } \epsilon \in (0, A - a],$$

where $K(A, a)$ denotes the parent's payoff of setting kid's cash-on-hand to $a_{0,s} = A$ given state $a_{0,g} = a$. Now, writing out the terms of K and using the fact that the marginal cost of savings drops as one becomes richer, we have for any $\epsilon \in (0, A - a]$:

$$\begin{aligned}
0 \geq & u\big(a - A + \epsilon\big) - u(a - A) + P_{0,s}(A - \epsilon) - P_{0,s}(A) \\
> & u\big(a + \delta - A + \epsilon)\big) - u(a + \delta - A) + P_{0,s}(A - \epsilon) - P_{0,s}(A) \\
= & K(A - \epsilon; a + \delta) - K(A; a + \delta)
\end{aligned}$$

for any $\delta > 0$. As in the proof for Lemma 4, the strict inequality is justified since $u(\cdot)$ is strictly concave by Ass. 1 and since $\delta > 0$. From the above it follows that $K(A - \epsilon; a + \delta) < K(A; a + \delta)$, i.e. $A - \epsilon$ is not optimal for state $a + \delta$, which completes the proof. ■

Proof. (Proposition 2.3) The parent's payoff from setting $A \geq x_{0,s}^{aut}$ given state a in the autarky region is $K(A;a) = u(y_0^p + a - A) + \alpha V_{0,s}^{aut}(A) + \beta u(y_1^p)$. Applying the Envelope Theorem to the child's value in autarky, the derivative of this function is given by

$$K_A(A;a) = -u'(\underbrace{y_0^p + a - A}_{=c_0^p}) + \alpha u'(\underbrace{A - A_{1,y}^{aut}(A)/R}_{=c_0^{k,aut}(A)}), \tag{22}$$

where the first term captures the marginal cost of giving (which is increasing in the gift A) and the second captures its marginal benefit (which is decreasing in the gift A since the child's autarkic problem is a standard savings problem in which consumption increases in initial assets).

Case 1: First, note that the marginal payoff of giving zero gifts, define it as $K_A^0(a) = K_A(a;a)$, is decreasing in a, i.e. the function $K_A^0(\cdot)$ is decreasing for $a \geq x_{0,s}^{aut}$ since $c_0^{k,aut}(\cdot)$ is a strictly increasing function. Now, the condition for Case 1 implies $K_A^0(x_{0,s}^{aut}) \leq 0$, which then means that $K_A^0(a) < 0$ for any $a > x_{0,s}^{aut}$, i.e. the marginal benefit of giving the first gift dollar is already negative. Since $K_A(A;a)$ is decreasing in A, the marginal benefit of giving must then be negative for all feasible $A > x_{0,s}^{aut}$, which directly implies that any gift to autarky must be a corner solution as described in the proposition.

Case 2: Conversely, if the condition for Case 2 holds, then for pairs (A, a) close to $(x_{0,s}^{aut}, x_{0,s}^{aut})$, we have $K_A(A, a) > 0$ by continuity of $u'(\cdot)$ and of $c_0^{k,aut}(\cdot)$ – recall again that the autarkic problem is a standard savings problem. Hence, for $\epsilon_1 > 0$ small enough a shot to autarky must occur by Prop. 2.2 and we have $K_A(x_{0,s}^{aut}, a) > 0$ for all $a \in (x_{0,s}^{aut} - \epsilon_1, x_{0,s}^{aut})$. Thus the shot must go into the autarky region. Second, for $\epsilon_2 > 0$ small enough we have $K_A(a;a) > 0$ for all $a \in [x_{0,s}^{aut}, x_{0,s}^{aut} + \epsilon_2)$, which implies that a positive gift is given. This concludes the proof of the claims in Case 2.

Sufficient condition for point landings. To show the last claim in the proposition, observe that

$$\alpha u'(x_{0,s}^{aut} - A_{1,y}(x_{0,s}^{aut})/R) = \alpha\beta R u'(A_{1,y}(x_{0,s}^{aut}) + y_1^k) \leq \beta R u'(y_1^p), \tag{23}$$

where the equality uses the child's Euler Equation for autarkic savings and the inequality uses the parent's first-order condition for gifts in the final period (which must be zero in autarky). If $u'(y_0^p) \geq \beta R u'(y_1^p)$, then (23) implies $u'(y_0^p) \geq \alpha u'(x_{0,s}^{aut} - A_{1,y}(x_{0,s}^{aut})/R)$ and thus $K_A(x_{0,s}^{aut}; x_{0,s}^{aut}) \leq 0$, i.e. the parent's marginal payoff from giving to the child when starting at the autarky threshold is negative (or zero), which is precisely the condition needed to guarantee that Case 1 occurs. ■

B Closed-form solutions for power utility

This appendix shows the solution for the power-utility case, i.e. invoking Assumption 2.

B.1 Auxiliary problems

We will first define the auxiliary problems. We will define these problems omitting the non-negativity constraints on gifts and savings in order to find relatively simple closed-form solutions for the unconstrained maximizers. For the case of power utility, all of the auxiliary problems are differentiable and concave and the resulting unconstrained policies are affine functions (which is a result of homotheticity of preferences). In the actual game, the auxiliary problems give us the equilibrium policy functions within certain regimes; they also useful to find the thresholds between regimes.

Dictator setting. Consider the setting in which the parent dictates the consumption allocation in the final period. In the savings stage, recall that we assume that the child can borrow against future family wealth to get rid of corner solutions.

Final period. In the final period, we can obtain the parent's dictator policy from (4) to obtain the policy

$$A_{1,s}^{dict}(a_{1,g}) = MPG^{sf}(y_1^p + a_{1,g}), \tag{24}$$

$$\text{where} \quad MPG^{sf} = \alpha^{1/\gamma}/(1+\alpha^{1/\gamma}), \quad a_{1,g} = y_1^k + a_{1,y}. \tag{25}$$

Here, MPG^{sf} is the marginal propensity to give when spoon-feeding, which will show up in the initial period as well.

Savings stage. Given the parent's decision rule in the final period, the child's problem in the savings stage is given by

$$\begin{aligned} &\max_{c_0^{k,dict}, c_1^{k,dict}, a_{1,y}} \left\{ \frac{(c_0^{k,dict})^{1-\gamma}}{1-\gamma} + \beta \frac{(c_1^{k,dict})^{1-\gamma}}{1-\gamma} \right\} \\ \text{s.t} \quad & c_0^{k,dict} + \frac{a_{1,y}}{R} = a_{0,s} \\ & c_1^{k,dict} = MPG^{sf}(y_1^p + y_1^k + a_{1,y}) \end{aligned}$$

A key insight is now that we can turn this into a standard savings problem with a modified interest rate and endowments. Denote $\tilde{y}_1^k = MPG^{sf}(y^p + y^k)$, $\tilde{a}_{1,y} = MPG^{sf} a_{1,y}$ and $\tilde{R} = MPG^{sf} R$ to write the child's problem equivalently as

$$\begin{aligned} &\max_{c_0^{k,dict}, c_1^{k,dict}, \tilde{a}_{1,y}} \left\{ \frac{(c_0^{k,dict})^{1-\gamma}}{1-\gamma} + \beta \frac{(c_1^{k,dict})^{1-\gamma}}{1-\gamma} \right\} \\ \text{s.t} \quad & c_0^{k,dict} + \frac{\tilde{a}_{1,y}}{\tilde{R}} = a_{0,s} \\ & c_1^{k,dict} = \tilde{y}_1^k + \tilde{a}_{1,y}, \end{aligned}$$

with solution

$$\begin{aligned}
c_0^{k,dict} &= \underbrace{(1+\beta^{1/\gamma}\tilde{R}^{(1-\gamma)/\gamma})^{-1}}_{=MPC^{dict}}\tilde{W}_0 \\
c_1^{k,dict} &= (1-MPC^{dict})\tilde{R}\tilde{W}_0 \\
A_{1,y}^{dict}(a_{0,s}) &= -MPC^{dict}\cdot(y_1^p+y_1^k)+(1-MPC^{dict})\cdot Ra_{0,s},
\end{aligned} \tag{26}$$

where we define $\tilde{W}_0 = a_{0,s} + \frac{\tilde{y}_1^k}{\tilde{R}}$ to be the present value of the (modified) endowment. The child's value function in the savings stage of the dictator setting is given by

$$V_{0,s}^{dict}(a_{0,s}) = \frac{(c_0^{k,dict})^{1-\gamma}}{1-\gamma} + \beta\frac{(c_1^{k,dict})^{1-\gamma}}{1-\gamma} \tag{27}$$

Initial gifts. The parent's gift-giving problem at $t=0$, knowing that the dictator game will ensue, is

$$P_{0,g}^{dict}(a) \equiv \max_{A\in[0,a+y_0^p]}\left\{u(a+y_0^p-A)+P_{0,s}^{dict}(A)\right\}, \tag{28}$$

where $P_{0,s}^{dict}(a)$ is the parent's value entering the child's savings stage described before. Denote the policy correspondence that solves this problem by $A_{0,s}^{dict}(a)$. Since the child's savings function is affine, this is a concave problem that we can solve in closed form as

$$\begin{aligned}
A_{0,s}^{dict}(a) &= \frac{B(a+y_0^p)-R^{-1}(y_1^p+y_1^k)}{1+B}, \\
\text{where}\quad B &= \left[\alpha(MPC^{dict})^{1-\gamma}+\beta R^{1-\gamma}(1-MPG^{sf})^{-\gamma}(1-MPC^{dict})^{1-\gamma}\right]^{1/\gamma}.
\end{aligned} \tag{29}$$

Autarkic setting. Recall that in autarky, we force final-period gifts to be zero. Again, we let the child borrow against its future endowment to guarantee interior solutions.

Savings stage. For the child, we have again a standard two-period savings problem with solution

$$\begin{aligned}
c_0^{k,aut} &= \underbrace{(1+\beta^{1/\gamma}R^{(1-\gamma)/\gamma})^{-1}}_{=MPC^{aut}}W_0 \\
c_1^{k,aut} &= (1-MPC^{aut})RW_0 \\
A_{1,y}^{aut}(a_{0,s}) &= -MPC^{aut}\cdot y_1^k+(1-MPC^{aut})\cdot Ra_{0,s}
\end{aligned} \tag{30}$$

where $W_0 = a_{0,s} + \frac{y_1^k}{R}$ is the present value of the endowment. The child's value function in the savings stage of the autarkic setting is given by

$$V_{0,s}^{aut}(a_{0,s}) = \frac{(c_0^{k,aut})^{1-\gamma}}{1-\gamma} + \beta \frac{(c_1^{k,aut})^{1-\gamma}}{1-\gamma} \tag{31}$$

Initial gifts. The parent's problem at $t = 0$ knowing that the autarkic allocation will ensue is given by

$$P_{0,g}^{aut}(a) \equiv \max_{A \in [0, a+y_0^p]} \left\{ u(a + y_0^p - A) + \alpha V_{0,s}^{aut}(A) \right\} + \beta u(y^p), \tag{32}$$

with the maximizer denoted by $A_{0,s}^{aut}(a)$. Note that this is a concave problem (the function inside the curly brackets is concave). Algebra gives us the policy

$$A_{0,s}^{aut}(a) = \frac{\alpha^{1/\gamma}}{MPC^{aut} + \alpha^{1/\gamma}}(a + y_0^p) - \frac{MPC^{aut}}{MPC^{aut} + \alpha^{1/\gamma}} \frac{y_1^k}{R}. \tag{33}$$

Constrained setting. Finally, consider a setting in which we force the child to consume all of its cash-on-hand in the savings stage at $t = 0$. Given child cash-on-hand a entering the game the parent's *constrained problem* is given by

$$P_{0,g}^{cd}(a) \equiv \max_{A \in [0, a+y_0^p]} \left\{ u(a + y_0^p - A) + \alpha u(A) \right\} + \beta P_{1,y}(0). \tag{34}$$

Denote the optimal policy in this problem by $A_{0,s}^{cd}(a)$. Note here that in the second period, we always have the same allocation: the one that ensues when the child enters with zero savings. This is essentially a static altruism problem and the resulting allocation here is the spoon-feeding policy that we already obtained in the final period; for power utility we have

$$A_{0,s}^{cd}(a) = MPG^{sf}(a + y_0^p), \tag{35}$$

where we recall that $MPG^{sf} = \alpha^{1/\gamma}/(1 + \alpha^{1/\gamma})$ is the marginal propensity to give under spoon-feeding. The child's value function in the savings stage of the constrained setting is given by

$$V_{0,s}^{cd}(a_{0,s}) = \frac{(A_{0,s}^{cd}(a_{0,s}))^{1-\gamma}}{1-\gamma} + \beta \frac{(MPG^{sf}(y_1^p + y_1^k))^{1-\gamma}}{1-\gamma} \tag{36}$$

B.2 Solving the actual game

Final period. In the final-period's gift-giving stage, we find the threshold between the dictator and autarky regimes from Eq. (7) as

$$x_{1,g}^{aut} = \alpha^{1/\gamma} y_1^p. \tag{37}$$

Using Eq. (6) and (24), the equilibrium gift-giving policy is thus piecewise linear and given by

$$A_{1,s}(a_{1,g}) = \max\{MPG^{sf}(y_1^p + a_{1,g}), a_{1,g}\}. \tag{38}$$

Savings stage. We can obtain the threshold $x_{0,s}^{cd}$ where the child's dictator savings become positive by solving $A_{1,y}(x_{0,s}^{cd}) = 0$. Using Eq. (26), this gives us

$$x_{0,s}^{cd} = (MPC^{dict}/(1 - MPC^{dict}))(y_1^p + y_1^k)/R. \tag{39}$$

We obtain the threshold at which autarky is entered, $x_{0,s}^{aut}$, by solving

$$\begin{aligned} \mathbb{I}(a < x_{0,s}^{cd})V_{0,s}^{cd}(a) + \mathbb{I}(a \geq x_{0,s}^{cd})V_{0,s}^{dict}(a) = V_{0,s}^{aut}(a), \quad a \in [\underline{x}, \bar{x}], \\ \text{where} \quad \underline{x} = \left(x_{1,y}^{aut} + MPC^{aut}y_1^k\right)/R(1 - MPC^{aut}), \\ \bar{x} = \left(x_{1,y}^{aut} + MPC^{dict}(y_1^p + y_1^k)\right)/R(1 - MPC^{dict}), \end{aligned} \tag{40}$$

where $x_{0,s}^{cd}$ is given by Eq. (39), $V_{0,s}^{cd}(a)$ is given by Eq. (36), $V_{0,s}^{dict}(a)$ is given by Eq. (27), and $V_{0,s}^{aut}(a)$ is given by Eq. (31). The optimal savings policy is then given by

$$A_{1,y}(a) = \max\left\{0, \mathbb{I}(a < x_{0,s}^{aut})A_{1,y}^{dict}(a) + \mathbb{I}(a \geq x_{0,s}^{aut})A_{1,y}^{aut}(a)\right\}$$

where $x_{0,s}^{aut}$ satisfies Eq. (40), $A_{1,y}^{dict}(a)$ is given by Eq. (26) and $A_{1,y}^{aut}(a)$ by Eq. (30).

Initial gifts. This can be done by first-order conditions since the problems are concave. We can then compare the values coming from the three sub-problems. To maximize a concave function on an interval, we have to pick (i) the lower boundary if the unconstrained maximizer falls to the left of the feasible set (this can be either giving zero gifts or staying at the left corner of the region under consideration), (ii) picking the unconstrained solution if it falls inside the feasible interval, (iii) the upper boundary if the unconstrained maximizer falls to the right of the feasible set.

Switch from constrained to dictator setting. Mathematically, we get the somewhat ugly (but closed-form) expressions

$$
\begin{aligned}
\tilde{A}_{0,s}^{cd}(a) &= \arg \max_{A \in [a, \min\{x_{0,s}^{cd}, a+y_0^p\}]} K(A,a) && (41)\\
&= \max\left\{a, \min\left\{A_{0,s}^{cd}(a), x_{0,s}^{cd}\right\}\right\} && \text{defined for } a < x_{0,s}^{cd}, \quad (42)\\
\tilde{A}_{0,s}^{dict}(a) &= \arg \max_{A \in [\max\{a, x_{0,s}^{cd}\}, \min\{x_{0,s}^{aut}, a+y_0^p\}]} K(A,a) && (43)\\
&= \max\left\{a, x_{0,s}^{cd}, \min\left\{A_{0,s}^{dict}(a), x_{0,s}^{aut}\right\}\right\} && \text{defined for } x_{0,s}^{cd} - y_0^p < a < x_{0,s}^{aut}, \quad (44)\\
\tilde{A}_{0,s}^{aut}(a) &= \arg \max_{A \in [\max\{a, x_{0,s}^{aut}\}, a+y_0^p]} K(A,a) && (45)\\
&= \max\left\{a, x_{0,s}^{aut}, A_{0,s}^{aut}(a)\right\} && \text{defined for } x_{0,s}^{aut} - y_0^p < a, \quad (46)
\end{aligned}
$$

where the parent's payoff function K is given by Eq. (14), $A_{0,s}^{cd}(a)$ by Eq. (35), $A_{0,s}^{dict}(a)$ by Eq. (29), $A_{0,s}^{aut}(a)$ by Eq. (33), $x_{0,s}^{cd}$ by Eq. (39), and $x_{0,s}^{aut}$ satisfies Eq. (40). We note that the corner solution that the parent gives all of $a + y_0^p$ to the child and consumes zero is irrelevant since the Inada condition ensures that this is sub-optimal (and thus the unconstrained maximizer always lies below this value).

Now, we can find the cut-off $x_{0,g}^{cd}$ where the dictator-savings region starts to dominate the constrained region by solving

$$
\underbrace{K(\tilde{A}_{0,s}^{cd}(a), a) - K(\tilde{A}_{0,s}^{dict}(a), a)}_{\equiv X_{cd}(a)} = 0 \qquad \text{for } a \in [\max\{x_{0,s}^{cd} - y_0^p, 0\}, x_{0,s}^{cd}]. \tag{47}
$$

Note that the function $X_{cd}(\cdot)$ is continuous since (with power utility) it is the sum of two continuous functions. We also expect it to be decreasing, although this cannot be shown. Most importantly, the function X_{cd} can cross zero only once on the relevant range, as is implied by Lemma 5.

If at $a = 0$ a shot to the dictator-savings region is already feasible, it may be that (i) Eq.(47) is already negative (or zero) at $a = 0$, which should be ruled out before solving. Also, it may be that (ii) Eq.(47) is zero at the upper end $a = x_{0,s}^{cd}$, in which case this is the switch to the dictator-savings region. If none of (i) or (ii) is true, then $x_{0,g}^{cd}$ can be found by finding the unique root of $X_{cd}(a)$ on the range $[\max\{x_{0,s}^{cd} - y^p, 0\}, x_{0,s}^{cd}]$.

Switch to autarkic setting. Denote the value from the better out of CD and DICT as

$$
P_{0,g}^{low}(a) = \mathbb{I}(a < x_{0,g}^{cd}) K(\tilde{A}_{0,s}^{cd}(a), a) + \mathbb{I}(a \geq x_{0,g}^{cd}) K(\tilde{A}_{0,s}^{dict}(a), a), \tag{48}
$$

which is well-defined for $a \in [0, x_{0,s}^{aut}]$ (i.e. until only autarky is a possibility) and continuous.

Finally, we can find the cut off where the switch from the "low" regime (CD or DICT) to autarky takes place. The solution is the number $x_{0,g}^{aut}$ that solves

$$\underbrace{P_{0,g}^{low}(a) - K(\tilde{A}_{0,s}^{aut}(a), a)}_{\equiv X_{aut}(a)} = 0 \qquad \text{for } a \in [\max\{x_{0,s}^{aut} - y_0^p, 0\}, x_{0,s}^{aut}]. \tag{49}$$

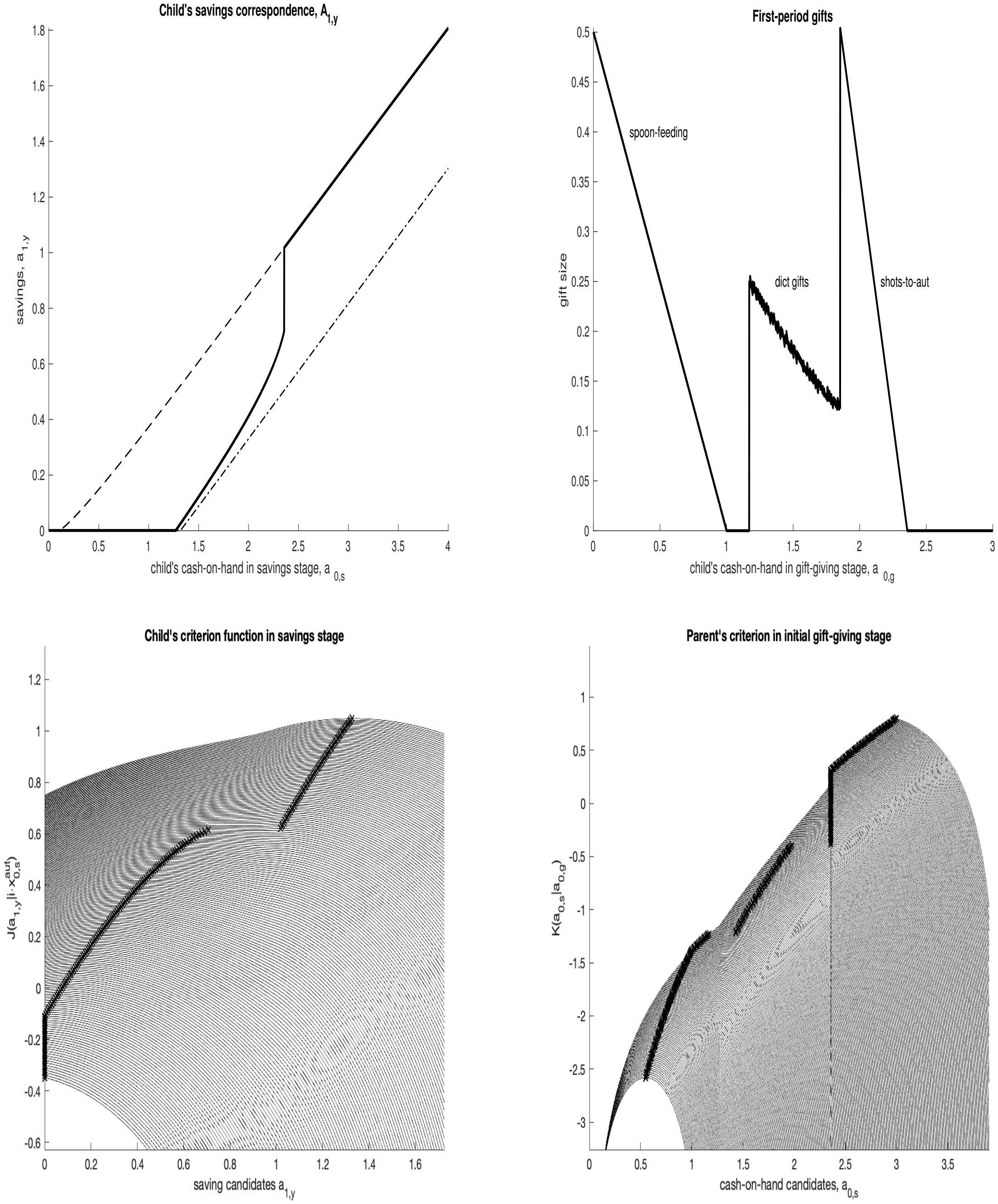
Again, there can only be one solution by Lemma 5 – once autarky becomes optimal, there cannot be a jump back to the low regime. For power utility, since $X_{aut}(\cdot)$ is the sum of two continuous functions, it is also continuous itself. If at $a = 0$ a shot to autarky is already feasible, it may be that (i) X_{aut} is already negative (or zero) at $a = 0$, which should be ruled out before solving – in this case autarky is *always* played.[25] Note that by Prop. 2.2, X_{aut} must switch to negative before reaching the upper end $a = x_{0,g}^{cd}$ – shots to autarky will be optimal once we get close enough to the autarky threshold since the parent's continuation value has an upward jump discontinuity. Thus, if case (i) does not apply, we find $x_{0,g}^{aut}$ by finding the root of $X_{aut}(a)$ on $[\max\{x_{0,s}^{aut} - y_0^p, 0\}, x_{0,s}^{aut}]$.

C Additional results

Figures 10 to 12 complement Section 4.2. They show the effects of increasing the child's expected endowment when child's income follows a log-normal distribution, $E(y^k) = 0.5, E(y^k) = 1$, and $E(y^k) = 1.5$, keeping the parent's deterministic endowment fixed at 1. The main observation is that shots-to-autarky disappear as the child's expected endowment increases which is similar to violating Condition 1 in the deterministic case in which case shots-to-autarky would also disappear because autarky is the only outcome in the final period.

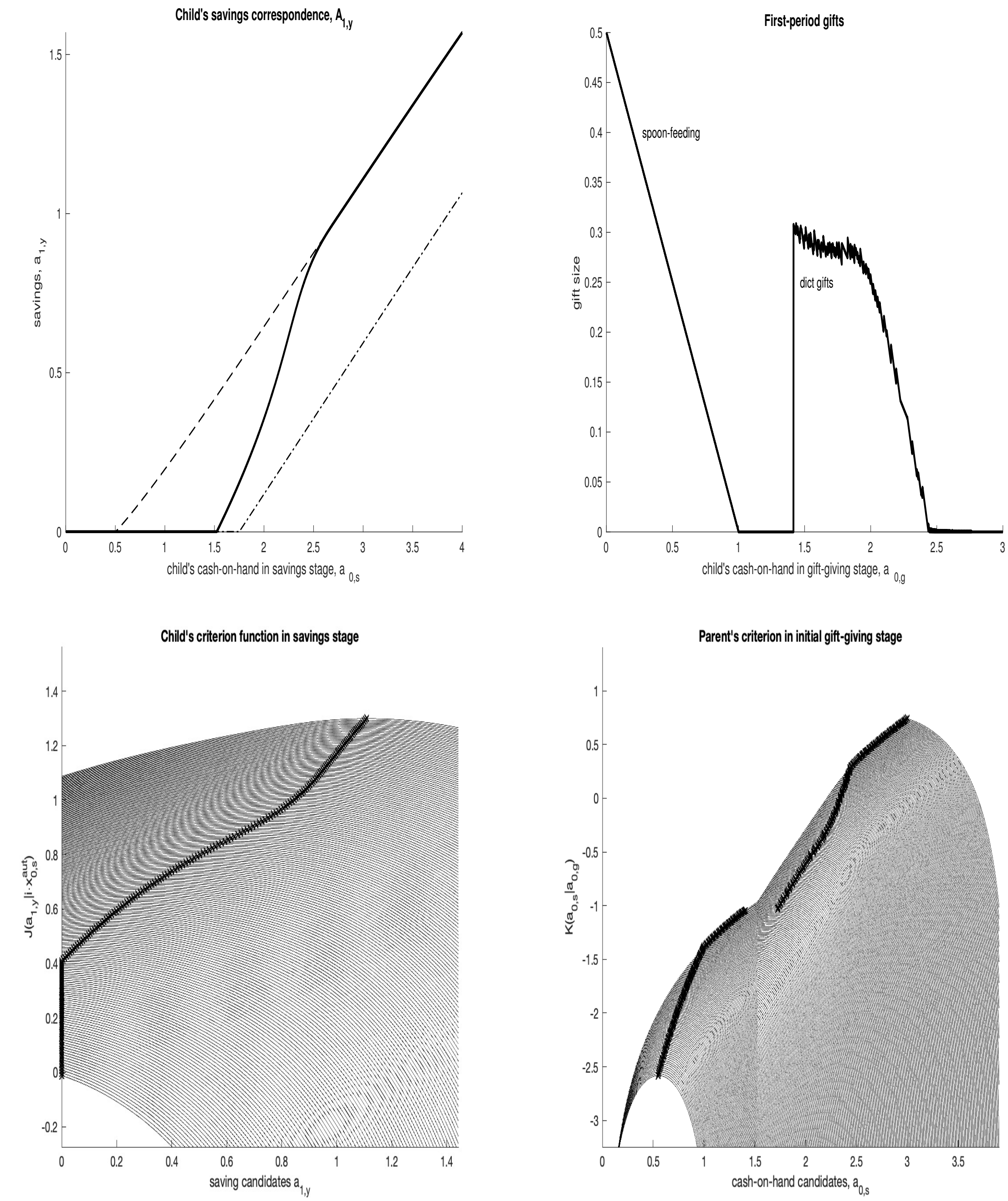
[25] Note that if the lower bound is $x_{0,s}^{aut} - y_0^p$, the parent would never prefer this lower bound by the Inada condition since parent consumption is zero in this case.

Figure 10: $t = 0$: continuous support, $E(y^k) = 0.5$



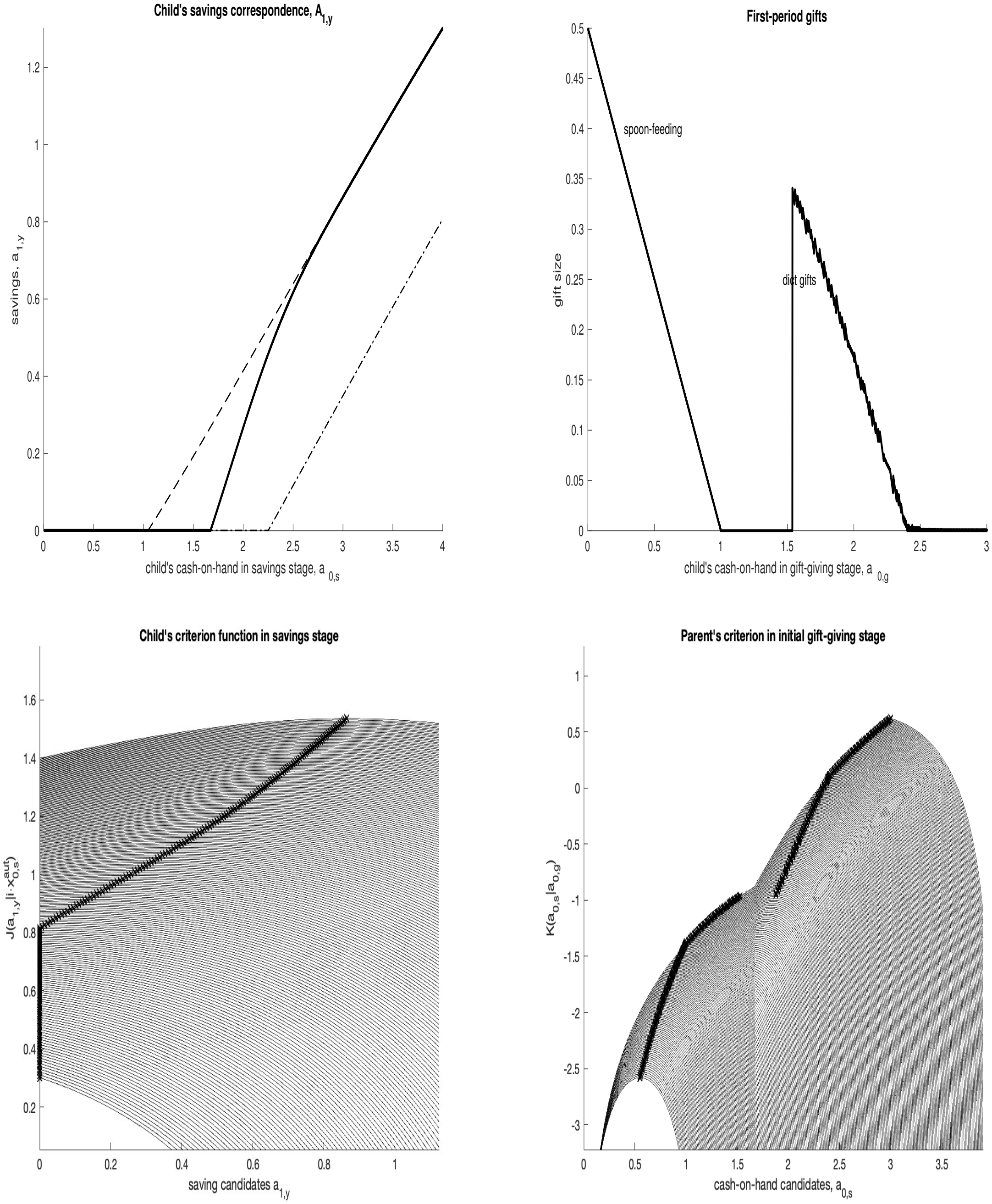
Left panel: Child's savings policy and criterion function in savings stage. Right panel: Parent's first-period gift policy and criterion function in gift-giving stage. Child's income in the income stage of the final period follows log-normal distribution with expected value equal to 0.5 and standard deviation equal to 1. Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and grid size $N = 5,000$.

Figure 11: $t = 0$: continuous support, $E(y^k) = 1$



Left panel: Child's savings policy and criterion function in savings stage. Right panel: Parent's first-period gift policy and criterion function in gift-giving stage. Child's income in the income stage of the final period follows log-normal distribution with expected value equal to 1 and standard deviation equal to 1. Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and grid size $N = 5,000$.

Figure 12: First period: $E(y^k) = 1.5$



Left panel: Child's savings policy and criterion function in savings stage. Right panel: Parent's first-period gift policy and criterion function in gift-giving stage. Child's income in the income stage of the final period follows log-normal distribution with expected value equal to 1.5 and standard deviation equal to 1. Utility is logarithmic. Parameters: $\alpha = \beta = R = 1$, $y^p = 1$, and grid size $N = 5,000$.