Online Appendix to: A Dynamic Model of Altruistically-Motivated Transfers

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Abstract

In this online appendix we provide proofs and additional results to the paper "A Dynamic Model of Altruistically-Motivated Transfers".

Contents

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1 His equations

Throughout the paper, we have only stated equations for her (player 1). For the reader's convenience, this section provides the mirror-symmetric versions for him (player 2).

His no-borrowing constraint is $k'_t \geq 0$. When broke, his transfers are constrained: $k'_t = 0$ implies $g'_t = 0$. For his consumption when broke, analogous to 1 in the paper, we have

$$ {c'}^*_t = \begin{cases} c'_t & \text{if } k'_t > 0 \text{ or } g_{mt} > 0 \\ \min\left\{c'_t, g_{ft}\right\} & \text{otherwise.} \end{cases} $$

His HJB in k-k'-space is, analogous to equation (15) in the paper:

$$ \rho v' = \max_{c',g'} \left\{ u(c') + \alpha' u(c) + (rk' - c' - g' + g)v'_{k'} + (rk - c - g + g')v'_k \right\}. $$

The FOC is $u_c(c') = v'_{k'}$ and the Euler equation is, analogous to (16) in the paper:

$$\frac{d}{dt}\big[u_c(c_t)\big] = (\rho - r)u_c(c') + \big[v'_k - \alpha' u_c(c)\big]c_{k'} + \big[v'_k - u_c(c')\big]g_{k'}.$$

In P-K-space his HJB is, analogous to (25) in the paper:

$$\begin{aligned}\rho V' =& \alpha' \ln C - C\frac{1+\alpha'}{\rho} - C(1-P)V'_P - GV'_P + \\ &+ \max_{C'\geq 0}\Big\{\ln C' - C'\frac{1+\alpha'}{\rho} + C'PV'_P\Big\} + \max_{G'\geq 0}\{G'V'_P\}.\end{aligned}$$

The corresponding FOC is, analogous to (28) in the paper:

$$\frac{1}{C'} = \frac{1+\alpha'}{\rho} - PV'_P,$$

and the Euler equation is, analogous to (29) in the paper:

$$\begin{aligned}\frac{d}{dt}V'_P(T) &= \big[PC' - (1-P)C - G + G'\big]V'_{PP} = \\ &= \big[\rho - C - C'\big]V'_P + \left[\frac{1}{C'} - \frac{\alpha'}{C} + V'_P\right]C_P + G_P V'_P.\end{aligned} \tag{1}$$

His value function in a SS-region is, analogous to equation (A.4) in the paper:

$$\rho {V'}^{SS} = (1+\alpha')(\ln\rho - 1) + \ln(1-P) + \alpha' \ln P.$$

The ODE for his transfers in a FT-region is, analogous to (A.6) in the paper:

$$G'_P = \frac{\alpha'}{1+\alpha'}\rho - C. \tag{2}$$

His value function in a WP region is, analogous to (A.7) in the paper:

$$\rho {V'}^{WP} = \ln C'_{WP} + \alpha' \ln C_{WP} - (C'_{WP} + C_{WP})\frac{1+\alpha'}{\rho}.$$

2 Our equilibrium concept versus the literature

2.1 Discontinuous strategies: a fundamental issue in differential games

Building on the discussion in Fudenberg & Tirole's (1993) textbook ("Game Theory", p.525), we first review a fundamental problem in the specification of differential games. The conventional approach builds on the theory of ordinary differential equations (ODEs) and optimal-control theory. The fundamental existence and uniqueness result for ODEs requires the function $f(\cdot)$ in the following differential equation to be Lipschitz-continuous:

$$\dot{x}_t = f(x_t),$$

where the dot denotes the time-derivative. In a 2-player differential game with Markov strategies, one specifies

$$\dot{x}_t = g(s_1(x_t), s_2(x_t)),$$

where s_i is the strategy by player i. Usually g is assumed to be a C^1 function. Then, in order for the ODE theorem to apply, we also need the strategies s_i to be C^1. So one restricts the strategy space to C^1 functions.

The problem arises when we want to check if a strategy s_1 is a best response to s_2. Even when s_2 is C^1, for the Pontryagin maximum principle to apply we need to allow the agent to choose s_1 in a space of functions that also contains piecewise-C^1 functions (which may have local jumps). What might go wrong is the following: s_1 may be the best response among all C^1 functions (and thus be consistent with equilibrium), but the maximum principle tells us it is not optimal because it is dominated by some $\tilde{s}_1$ that is discontinuous.

Is it then a good idea to extend the strategy space to piecewise C^1 functions? No, because then payoff functions are not well-defined any more. If the law of motion $g(s_1(\cdot), s_2(\cdot))$ is a discontinuous function of the state, then existence and uniqueness for the ODE is not ensured any more. So this does not work either.

2.2 Why the viscosity concept does not solve the problem

Viscosity solutions are the agreed-upon concept that tells us in which sense the Hamilton-Jacobi-Bellman equation (HJB) from optimal-control theory holds as a partial differential equation (PDE).[1] It becomes important when the value function has kinks. Even in simple control problems (i.e. one-player games) such situations can easily arise. We will first illustrate the viscosity principle in a simple control problem and then discuss why it does not solve the above-mentioned discontinuity issues in differential games.

2.2.1 A brief illustration of the viscosity principle in a control problem

Consider a traveler in the desert who obtains disutility of -1 for each unit of time she spends in the desert. $x \in [-1, 1]$ designates the traveler's location, and there is an oasis located at $x = -1$ and $x = 1$. The game is over when the traveler reaches an oasis. The traveler controls her speed and direction, but can travel at a maximum speed of 1, i.e. $|\dot{x}| \leq 1$. The solution to the traveler's problem is, of course, to set $\dot{x} = 1$ if $x > 0$ and $\dot{x} = -1$ if $x < 0$ and $\dot{x} \in \{-1, 1\}$ if $x = 0$, and the value function v will be given by the distance to the next oasis. But we will now follow standard control theory to see why we need the viscosity principle already in this seemingly innocent example.

Bellman's principle for the traveler is

$$v(x_t) = \max_{|\dot{x}| \leq 1} \{-1 \cdot \Delta t + v(x_{t+\Delta t})\} \quad \text{s.t.} \quad x_{t+\Delta t} = x_t + \dot{x}\Delta t,$$

[1]The seminal paper is Crandall & Lions (1983). An excellent tutorial for viscosity solutions is Bressan (2010), which we largely follow here.

from which we obtain the HJB

$$0 = -1 + \max_{|\dot{x}|\leq 1} \left\{ \dot{x} v_x \right\}.$$

The optimal policy given $v_x(x)$ is to set $\dot{x} = 1$ if $v_x \geq 0$ and $\dot{x} = -1$ if $v_x < 0$. This implies that v satisfies the following boundary-value problem:

$$0 = -1 + |v_x|, \qquad \text{s.t. } v(-1) = v(1) = 0. \tag{3}$$

This is a first-order ODE, and we are looking for a solution $v(x), x \in [-1, 1]$. Let us first try to find a solution v that is differentiable everywhere. Note that at a critical point x^* we would have $v_x(x^*) = 0$, which clearly does not satisfy (3). Thus v cannot have any local minimum or maximum on $(-1, 1)$. This leaves as the only possible solution $v(x) = 0$ for all x, which does not work either.

Let us now try a weaker concept and try functions that are differentiable *almost* everywhere and satisfy (3) whenever its derivative exists. It will turn out that this concept is too weak since it does not lead to any useful uniqueness results. (3) tells us that v must either have slope 1 or -1 on its smooth parts. But apart from the true value function, there are infinitely many other "saw-tooth" functions that change their slope more than once and satisfy the two boundary conditions. A simple example is a function that takes minima at $x \in \{-a, a\}$ and a local maximum at 0. If the traveler took this value function at face value for locally deciding where to go, he would walk towards the middle of the desert on $(-a, a)$, which is a lethal policy. Given the traveler's policy on $[-1, 1] \setminus 0$, the only sensible interpretation at 0 is that the traveler will stay at $x = 0$ forever and thus obtains $v = -\infty$, which contradicts the suggested solution. But our current solution concept does not give us any restriction at the point $x = 0$ that would rule out this nonsensical behavior.

Viscosity solutions will give us the correct restrictions at such points and leave the correct value function as the only solution to (3). The term viscosity solution comes from the fact that v can be obtained as a limit of smooth solutions to the PDE

$$F(x, v^\epsilon, \nabla v^\epsilon) = \epsilon \Delta v^\epsilon, \tag{4}$$

i.e. we have $v = \lim_{\epsilon \searrow 0} v^\epsilon$. $F(\cdot)$ gives the PDE we want to solve as a function of x, v and the gradient ∇v, which is in our case the right-hand side of (3). Δ denotes the Laplacian, which in one dimension equals the second derivative.[2] Adding second derivatives ensures a smooth solution to the PDE; the constant ϵ is related to the "thickness" of a fluid in physics applications of PDEs, thus the name "viscosity solution".

We can add noise to our traveler's problem to give an interpretation to the vanishing viscosity solutions in our example. Specifically, we add Brownian motion to the control problem above, which may be interpreted as random factors affecting the speed of travel. The Δt-problem is then

$$v(x_t) = \max_{|\dot{x}|\leq 1} \left\{ -1 \cdot \Delta t + E\left[v(x_{t+\Delta t}) \right] \right\} \quad \text{s.t.} \quad x_{t+\Delta t} = x_t + \dot{x}\Delta t + \epsilon \Delta B_t,$$

[2] See Bressan's tutorial p.11, Theorem 4.3.

where $\epsilon > 0$ is the strength of the stochastic disturbance. Using the Ito rule, we obtain the HJB

$$0 = -1 + \frac{\epsilon^2}{2} v_{xx} + \max_{|\dot{x}| \leq 1} \{\dot{x} v_x\}$$

Upon substituting the optimal policy – which is the same as in the deterministic case – into the HJB we obtain an ODE that is in the form of (4):

$$1 - |v_x| = \frac{\epsilon^2}{2} v_{xx},$$

to which a smooth solution must exist. At a critical point x^* we have that $v_x(x^*) = 0$. If x^* was a local maximum, then $v_{xx}(x^*) < 0$, which would contradict the HJB. But x^* may be a local minimum, since then $v_{xx}(x^*) > 0$, which is consistent with the HJB. Thus, v can only have local minima, and indeed there must be a unique (and thus global) minimum x^*. For $x < x^*$ we have $v_x < 0$, and for $x > x^*$ we have $v_x > 0$. The solutions are given by downward-facing U-shapes with $x^* = 0$ (by symmetry), the solutions getting spikier as ϵ gets small. We conclude that in the limit, the solution must be given by $v = -1 + |x|$. This is indeed the correct solution to our original control problem.[3]

2.2.2 Why the viscosity solution does not solve the discontinuity issues in differential games

We will now show in two examples why and how the viscosity approach fails to solve the issues that we face in our differential game.

Case 1: The law of motion is discontinuous. We extend the above setting of the traveler to two players. We may now think of two persons on a boat who are on a lake and want to reach the shore. Each of them has a paddle and can influence direction and speed of the boat. Given the players' controls $y \in [-1, 1]$ and $y' \in [-1, 1]$, the law of motion for the boat is

$$\dot{x}_t = \tfrac{1}{2} y_t + \tfrac{1}{2} y'_t.$$

As before, each agent obtains one unit of disutility for each unit of time spent on the lake. The efficient solution, in which both players cooperate in rowing to the closest shore, should be supported by any reasonable equilibrium concept. But we will see that in purely technical terms, the viscosity approach fails already in this simple example.

To see this, consider the control problem that player 1 faces when agent 2 plays the efficient equilibrium strategy. Let $y' = 1$ for $x \geq 0$ and $y' = -1$ for $x < 0$. The function F is then defined by

$$F(x, v_x, y) = -1 + v_x \cdot \begin{cases} \frac{1}{2} + \frac{1}{2} y & \text{if } x \geq 0, \\ -\frac{1}{2} + \frac{1}{2} y, & \text{if } x < 0. \end{cases}$$

[3] We refer the reader to Fleming & Soner (2006), p.60-61, for additional examples where the value function is not differentiable.

The theorems from viscosity theory do not apply because F is discontinuous in x. Furthermore, theorems from ODE theory cannot be applied either when F is not Lipschitz-continuous in x.[4]

Case 2: measure-valued controls. Consider the setting in our paper, and restrict attention to the case where players use homogeneous strategies. Then it is natural to conjecture that there exists an equilibrium where the poor player is lifted out of poverty by a mass transfer from the rich player and then both are self-sufficient ever after. Unfortunately, the viscosity concept does not deal with the case of measure-valued controls, so it doesn't allow us to make progress on this front either. With our concept, however, we are able to rule out this type of equilibrium when altruism is imperfect (see the Prodigal-Son Dilemma, Theorem 3 in the paper).

It is worthwhile to point out that a naive reading of the equilibrium definition in differential games can lead to trouble when it comes to measure-valued controls. One may argue as follows: a mass transfer implies that no time is spent in a mass-transfer region, so we should disregard consumption policies in such a region when calculating the player's value from equilibrium when the game starts off in such a region. The problem with this kind of reasoning is that it may allow for equilibria with threats in mass-transfer regions that are not credible.

To see this, consider the following example. It is plausible to conjecture an equilibrium with a SS-region between $\alpha'/(1+\alpha')$ and $1/(1+\alpha)$, which is the largest-possible range a SS-region can cover (see Lemma 7 in the paper). When the fraction of wealth owned by him is large, i.e. $0 \leq P < \alpha'/(1+\alpha')$, he provides a mass-transfer to her, filling up her bank account instantaneously so that $P = \alpha'/(1+\alpha')$. Vice versa, if the fraction she owns is large, i.e. $1/(1+\alpha) < P \leq 1$ she provides a mass transfer to him. Suppose that the transfer recipient's strategy is to set $C = 0$ in the region where (s)he receives a mass transfer. In order to find out if this is an equilibrium, let us check for potential deviations from the equilibrium strategies. Given her strategy to set $C = 0$ for $P < \alpha'/(1+\alpha')$, he has to give a mass transfer in order to ensure that no time is spent in that region, as otherwise his utility is $-\infty$. Given that the donor provides a mass transfer, $C = 0$ is indeed a best response for her if $P < \alpha'/(1+\alpha')$. So the suggested profile could indeed be considered an equilibrium under a naive reading of optimality.

However, we argue that such threats inside mass-transfer regions are not credible. In order to evaluate deviations for the potential donor it seems natural to consider what would happen if the economy remained inside the mass-transfer-region for a short period of time Δt. Then the recipient's threat $C = 0$

[4]See Olsder (2002) for a discussion in a deterministic setting; see Mannucci (2004) who deals with discontinuous policies in a stochastic setting; and again see Bressan's (2010) tutorial, theorem 10.1. One may think that the above game of two rowers may be solved by introducing noise into the problem in the spirit of vanishing viscosity solutions. But it turns out that, at least in this example, this leads to a non-standard boundary problem in stochastic differential equations since player 2's strategy should still be discontinuous. See Øksendal (2010), chapter 9, for more on boundary problems in stochastic settings.

is actually not credible. Our equilibrium concept does this by breaking the Δt-game into a transfer and a consumption stage and by requiring subgame-perfection in the latter.

2.3 How our concept deals with discontinuity

We have just seen how our equilibrium concept deals with determining best responses in the case of measure-valued controls. We will now see how it deals with discontinuity of policies and the associated kinks in value functions. Since our concept takes directional derivatives into account at boundaries, it is still well-defined at these points, even if the other player's strategy is discontinuous. It is straightforward to see that this equilibrium concept selects the optimal strategy and value function in the case of the desert wanderer (Section 2.2.1) just as the viscosity principle does. In the differential rowing game from Section 2.2.2, our equilibrium definition also tells us that the efficient strategy profile is an equilibrium: in the middle of the river, both agents optimally coordinate on going to the right. At all other points of the state space, it coincides with the optimality requirements from the standard HJB.

In these examples, there is no ambiguity of how to interpret the law of motion for x at the boundary $x = 0$ since the economy moves away from this point. It is similarly unproblematic when a boundary is such that the path of the state crosses the boundary: we can solve the ODE up to the boundary and then solve another ODE from the boundary onward.[5] If the equilibrium laws of motion are such, however, that the economy moves into a region from a boundary point but is then immediately repelled back towards the boundary, then we cannot solve for the path of the state using standard ODE techniques. We now present an example of how our equilibrium still makes reasonable predictions in such a case, and how we may interpret the equilibrium law of motion as "jiggering" around the boundary.

2.3.1 An example where the ODE for the state does not satisfy the Lipschitz condition

Consider the following game. A father is rowing with his son on a river. The boat's position on the river is denoted by $x \in [-1, 1]$: -1 and 1 are the shores, 0 is the middle. Both the father and the son have a paddle, but the father is stronger than the son. The law of motion is $\dot{x} = (1-\alpha)y^f + \alpha y^s$, where $y^i \in [-1, 1]$ is the control of agent i and $\alpha \in (0, \frac{1}{2})$ is the relative strength of the son. The father wants to keep the boat in the middle of the river because there is the risk of damaging the boat at the shores where the water gets shallow; his flow payoff is $u^f(x) = -|x|$. The son has no concerns about safety, he just wants action. His flow payoff is 0 if $\dot{x} \neq 0$, and -1 if $\dot{x} = 0$. Both players discount the future at rate ρ.

[5] The reasoning from Lemma 4 in the paper implies that policies on the boundary must equal the limiting policies inside the region where the economy moves, so that it is obvious how the ODE should be interpreted right at the boundary.

We will now show that the following strategies are an equilibrium according to our concept. Both players row towards the middle of the river whenever $x \neq 0$. When $x = 0$, the father sets $y^f = 0$ and the son sets $y^s = 1$. The equilibrium law of motion is thus

$$\dot{x} = \begin{cases} 1 & \text{if } x < 0 \\ \alpha & \text{if } x = 0 \\ -1 & \text{if } x > 0. \end{cases}$$

This law of motion has no solution as an ODE in the conventional sense at $x = 0$. However, it is reasonable to interpret the path as the boat zig-zagging (jiggering) around the middle of the river once it gets there. We now show how to construct the value functions and to verify that the proposed profile is an equilibrium according to our concept.

The son's equilibrium value function is obviously $v^s(x) = 0$. He is best-responding for all x since he ensures that there is action at all x. As for the father, we have the HJB

$$\rho v^f(x) = -|x| + \max_{y^f \in [-1,1]} \Big\{ \big[(1-\alpha)y^f + (2\alpha I_{x<0} - \alpha)\big] v_x^f(x) \Big\}, \tag{5}$$

where $v_x^f(x)$ is understood to be the directional derivative and $I_{x<0}$ is the indicator function for the set $[-1, 0)$. We will first obtain the father's value at the "steady state" $x = 0$. When cutting time into intervals of length Δt, we may interpret the law of motion as moving a distance $\alpha \Delta t$ to the right from 0 and then returning to 0 within a time interval of length $\tilde{\Delta t} = \alpha \Delta t$. This process repeats itself over and over. The father's value from this path is clearly lower-bounded by $\int_0^\infty e^{-\rho t}(-|\alpha \Delta t|)dt = -\frac{\alpha \Delta t}{\rho}$, since $-\alpha \Delta t$ is the worst flow payoff the father can have on the interval $[0, \alpha \Delta t]$. The upper bound for the father's value is 0. Taking the limit as $\Delta t \to 0$ the two bounds converge, and the only sensible value to assign is $v^f(0) = 0$.

We now use the equilibrium policies in the father's HJB (5) to obtain an ODE for v^f. v^f will obviously be decreasing on $[0, 1]$, so we have

$$v_x^f(x) = -x - \rho v^f(x) \qquad \text{for } x \in [0, 1]. \tag{6}$$

With the boundary condition $v^f(0) = 0$, this ODE can easily be solved for v^f on $[0, 1]$.[6] By symmetry, we obtain v^f on the range $[-1, 0)$ as $v^f(-x) = v^f(x)$. The ODE (6) also tells us that v^f is flat at $x = 0$, i.e. $v_x^f(0) = 0$. So $y^f = 0$ is indeed optimal for the father at $x = 0$, and the HJB (5) is fulfilled everywhere. The father is also best-responding, which proves that the suggested profile is an equilibrium according to our concept. We interpret the law of motion of the boat around $x = 0$ in equilibrium as the saw-tooth paths described above.

An interesting special case is the one without discounting ($\rho = 0$). The ODE (6) for v^f becomes $v_x^f = -x$, and the value function has the simple

[6] There is a closed form for v^f which we do not present since it does not add anything to the analysis.

quadratic form $v^f(x) = -\frac{1}{2}x^2$. Note that in this case, the construction of the value $v^f(0) = 0$ at the boundary does not go through any more, but the strategy profile still fulfills best responding, so it still constitutes an equilibrium according to our definition.

2.3.2 Constructing the value function at boundaries: policy-dependent flow utility and higher dimensions

The calculation of the value at the boundary, $v^f(0)$, was especially simple in the above game because the father's flow utility did not depend on policies (y^f, y^s) and thus converged to $u(0) = 0$. If flow utility depends on policies, things may become slightly more complicated. We suggest the following procedure for this case. Fix again a short time interval Δt during which the policies $y^f(0)$ and $y^s(0)$ are played and calculate the time interval $\tilde{\Delta t}$ it takes for the state to return to the boundary (in the above game: $\tilde{\Delta t} = \alpha\Delta t$). Then assign flow utility $u[0, y^f(0), y^s(0)]\Delta t + u[x_{\Delta t}, y^f(x_{\Delta t}), y^s(x_{\Delta t})]\tilde{\Delta t}$ over the time horizon $\Delta t + \tilde{\Delta t}$, where $x_{\Delta t} = 0 + \dot{x}_{|x=0}\Delta t$ is the state after having moved away from $x = 0$ over a time interval Δt. After taking limits as $\Delta t \to 0$ and integrating over $t = [0, \infty)$ will give a reasonable candidate value for $v^f(0)$.

Another feature that makes the above example simple is that the state is one-dimensional. With a higher-dimensional state, we may return to a different location, $x_{\Delta t+\tilde{\Delta t}}$, on the boundary after one zig-zag motion (i.e. after a time interval $\Delta t + \tilde{\Delta t}$). In this case, we suggest to write an HJB with flow utility $u[0, y^f(0), y^s(0)]\Delta t + u[x_{\Delta t}, y^f(x_{\Delta t}), y^s(x_{\Delta t})]\tilde{\Delta t}$, the directional derivative $\nabla_{(x_{\Delta t+\tilde{\Delta t}}-0)}v(0)$ and a law of motion $\dot{x} \propto (x_{\Delta t+\tilde{\Delta t}} - 0)$ that takes the economy along the boundary. For this approach to work it is essential, of course, that the boundary is sufficiently well-behaved.

3 Technical note on subgame-perfection

It is a hallmark of subgame perfection that the partial derivatives c'_k, $c_{k'}$, g'_k, $g_{k'}$ are present in the Euler equations. These derivatives tell us about the other player's "threats" in case one deviated from the equilibrium policy. These threats have to be credible in the sense of subgame perfection, i.e. agents' policies must be mutual best responses on these neighboring paths as well. But this implies that both agents' HJBs (and thus Euler equations) have to hold in an entire neighborhood of the path under consideration. Indeed, the equilibrium concept requires that both agents' HJBs be fulfilled for *every* point in the state space, so we have to find a solution for the system of PDEs given by his and her HJB on the entire k-k'-plane. This is also related to the fact that the usual classical calculus-of-variations arguments do not apply: we cannot construct a deviation from the optimal path that reverts to the optimal path, as becomes clear from Figure 4 in the paper.

Only in the special cases of $\alpha = \alpha' = 0$ and $\alpha = \alpha' = 1$ do the partial derivatives c'_k etc. disappear. Then, we can solve an ODE (and not a PDE) for consumption along the equilibrium path in the spirit of Pontryagin's maximum principle. In this special case, we do not have to take into account information from neighboring equilibrium paths.

4 Special cases

This section presents the commitment equilibrium and its proof as well as proofs related to the following modifications of our environment:

1. Transfers are ruled out (self-sufficiency, SS).
2. There are no property rights (wealth-pooling, WP).

All equations referenced in the proofs are understood to be those in the paper.

4.1 Commitment equilibrium

First, we have a brief look at a modification of our setting in which agents have the possibility to commit to future consumption and transfers. A *commitment strategy* is defined as a set of functions of time $B = \{C(t), G_f(t), G_m(t)\}$. It is understood that $c_t = C(t)K_t$ etc. and that $\{K_t, P_t\}$ follow the laws of motion given by (20) and (21) in the paper.[7]

Definition 1 (Commitment equilibrium) *A (homogeneous)* commitment equilibrium *is a pair of strategies* $B(t) = [C(t), G_f(t), G_m(t)]$ *and* $B'(t) = [C'(t), G'_m(t), G'_f(t)]$ *such that* B *maximizes her criterion* v_0 *at* $t = 0$ *given* B'*, and* B' *maximizes* v'_0 *at* $t = 0$ *given* B*.*

We now show that under commitment, an efficient equilibrium exists in which players give transfers only at $t = 0$ and are self-sufficient ever after:

Proposition 1 (Commitment) *The following strategies are a commitment equilibrium:*

- *She gives an initial mass transfer* $G_m(0) = \max\{P_{t=0} - \frac{1}{1+\alpha}, 0\}$*. She gives no transfers for all* $t > 0$ *and sets* $C(t) = \rho(P_{t=0} - G_m(0))$ *for all* t*.*
- *He gives an initial mass transfer* $G'_m(0) = \max\{\frac{\alpha'}{1+\alpha'} - P_{t=0}, 0\}$ *but gives no transfers for all* $t > 0$*. He sets* $C'(t) = \rho(1 - P_{t=0} + G'_m(0))$ *for all* t*.*

The equilibrium allocation is efficient for any initial distribution of wealth, $P_{t=0}$*. Furthermore, any efficient consumption allocation may be implemented by choosing an appropriate* $P_{t=0}$*.*

[7] We define realized consumption when she is broke as $c^*_t = C^*_t K_t$, where $C^*(t) = \min\{C(t), G'_f(t)\}$. She will thus never set $C(t) = 0$.

We note that in this equilibrium, the richer agent gives an initial transfer to implement her/his preferred allocation if the initial wealth share $P_{t=0}$ is outside the range $[P_0^*, P_1^*]$ spanned by the weight $\eta \in [0,1]$ in the Pareto solution (14) in the paper. This is also the type of equilibrium that obtains in a static altruism setting in which a transfer stage precedes the consumption stage. In essence, commitment removes the dynamic component from the game.

Proof: We first prove our claims about efficiency of the equilibrium. First, it is obvious that the equilibrium for $P_{t=0} \in (P_0^*, P_1^*)$ map one-to-one to the solutions of the planner's problem indexed by $\eta \in (0,1)$ as given by (14) in the paper. For all other initial conditions, the mass transfers induce efficient allocations with Pareto weights $\eta \in \{0,1\}$. So the equilibrium is efficient, and any efficient allocation is an equilibrium for some $P_{t=0}$.

It remains to show that the proposed strategies are indeed a commitment equilibrium. First, note that if $P_{t=0} \geq P_1^*$ her strategy is definitely a best response since it attains her preferred allocation among all feasible allocations since it corresponds to the solution of the Pareto problem with $\eta = 1$.

Consider now the case $P_{t=0} < P_1^*$. Define her lifetime wealth as $w_0 = k_0 + \int_0^\infty e^{-rt}(g_t' - g_t)dt$. The value of the equilibrium allocation for $P_{t=0}$ then gives us an upper bound for the payoff that is attainable by any $\{c_t, c_t'\}$ for a given lifetime wealth $w_0 = P_{t=0}K_0$, as is clear from the Pareto problem.[8] Define this upper bound as $\bar{v}(w_0)$. Now note that any positive transfer by her would reduce lifetime wealth to $\tilde{w}_0 < w_0$, and the payoff of any such deviation would be upper-bounded by $\bar{v}(\tilde{w}_0) < \bar{v}(w_0)$. Thus giving positive transfers must be sub-optimal if $P < P_1^*$. Given that she gives no transfers, her consumption strategy is then obviously optimal by the above arguments.

By the same arguments, his equilibrium strategy is a also a best response, which concludes the proof. ■

4.2 Proof for Proposition 3 (SS equilibrium)

Proof: Consider first the case where at least one player is altruistic, say $\alpha > 0$. Then, she would obtain a value of minus infinity under SS when he is broke since $C_{SS}' = 0$. So she should respond with a mass transfer at this point, which shows that self-sufficiency is not an equilibrium when $\alpha + \alpha' > 0$.

Consider now the case $\alpha = \alpha' = 0$. Given that the other player never gives transfers, the best response is obviously to respond with zero transfers and follow the consumption rule of an SS saver. Thus the SS policies constitute an equilibrium, which is clearly efficient.

Finally, we have to establish that the SS-policies are the unique equilibrium if $\alpha = \alpha' = 0$. Note that the SS policies are feasible for any initial conditions, so the value for each player in equilibrium is lower-bounded by the SS value for both players. If there was a profitable deviation from the SS strategy for her, then the time-0 value of this deviation would exceed the value from the

[8]Note that there might be additional restrictions on consumption paths coming from borrowing constraints that are due to the timing of transfers which may make it infeasible to reach this upper bound.

SS equilibrium. But the value to him would still have to be at least the value from SS since it is a lower bound. Thus a profitable deviation by her must lead to a Pareto improvement. But this is impossible, since any SS allocation is Pareto-efficient when $\alpha = \alpha' = 0$ (see Section 2.3 in the paper). ■

First, it should be pointed out that this proof also applies to non-Markovian strategies, so there are no tit-for-tat strategies either that could implement non-SS equilibria under selfishness. Second, it is interesting to note that the proof for uniqueness of the SS equilibrium does not go through in a stochastic setting, when potential gains from mutual insurance arise: then there are possible Pareto improvements from risk sharing which can raise value functions above the SS levels.

4.3 Wealth-pooling (WP)

4.3.1 Solving the WP model

Consider the following WP game: the physical environment is as specified in the paper, but players have no property rights, i.e. they consume out of a common asset pool K. We consider homogeneous strategies, i.e. players' strategies are characterized by numbers C_{WP} and C'_{WP}, which give us the consumption rates out of the common asset stock. The law of motion for K is then

$$\dot{K}_t = rK_t - C_{WP}K_t - C'_{WP}K_t.$$

Let $V^{WP}(K)$ be her value function. Her HJB then reads:

$$\rho V^{WP} = \alpha \ln(C'_{WP}K) + (r - C'_{WP}K)V_K^{WP} + \max_{C \geq 0} \left\{ \ln(CK) - CKV_K^{WP} \right\}.$$

The first-order condition (FOC) for C is

$$\frac{1}{C_{WP}} = V_K^{WP} K.$$

Taking derivatives of her HJB with respect to K gives us the Euler equation (EE)

$$\rho V_K^{WP} = \frac{1+\alpha}{K} + (r - C_{WP} - C'_{WP})(V_K^{WP} + KV_{KK}^{WP}).$$

Taking the derivative of the FOC in K shows that $KV_{KK}^{WP} = -V_K^{WP}$, so the last term in the EE vanishes. Using again the FOC in the EE then shows that WP consumption rates are as claimed in the paper, i.e.

$$C_{WP} = \frac{\rho}{1+\alpha}, \qquad C'_{WP} = \frac{\rho}{1+\alpha'},$$

where we note that his consumption rate C'_{WP} may be derived in a way entirely analogous to hers.

4.3.2 Proof for Proposition 4 (WP equilibrium)

Proof: We first show point 1 of the proposition.

Consider first the case $\alpha = \alpha' = 1$. For any pair of transfer strategies that make the wealth-pooling (consumption) allocation feasible, both agents are clearly best-responding since they obtain their globally-preferred allocation. So the wealth-pooling strategies are an equilibrium.

Second, we turn to the case $\alpha + \alpha' < 2$. Consider the situation where the more-altruistic agent is bankrupt: without loss of generality, let $P = 0$ and $\alpha > \alpha'$. Then given that her strategy is $C(0) = C_{WP}$ he should set transfers to $G_f(0) = \alpha' C'_{WP}$, which lowers her consumption to $\frac{\alpha'}{1+\alpha'} < C_{WP} = \frac{1}{1+\alpha}K$ (the inequality holds since $\frac{\alpha'}{1+\alpha'} < \frac{1}{2} \leq \frac{1}{1+\alpha}$). This implements his globally preferred allocation and thus dominates the wealth-pooling outcome. So we have found a profitable deviation, and thus wealth-pooling cannot be sustained in equilibrium.

We now proceed to show point 2 of the proposition. Let $\tilde{V}(P, K)$ be her value and $\tilde{V}'$ his value. Since the criteria by which players rank allocations coincide in the case of perfect altruism, it follows that $\tilde{V}(P, K) = \tilde{V}'(P, K)$ for all P. Order-0 optimality implies that $\tilde{V}$ is weakly increasing in P and $\tilde{V}'$ is weakly decreasing in P, by the same argument as in Proposition 2 in the paper. So it must be that $\tilde{V}$ is invariant in P for fixed K. We may thus drop the argument P and write a value function $\tilde{V}^{pa}(K)$ only in K. The HJB (equation (25) in the paper) becomes

$$\rho \tilde{V}^{pa}(K) = \max_C \{\ln C + \ln C' + 2 \ln K + (r - C - C')K\tilde{V}^{pa}_K\}.$$

For optimal policies we obviously have $C(K) = C'(K)$ for all K since his HJB is the same. We may thus write

$$\rho \tilde{V}^{pa}(K) = \underbrace{\max_C \{2 \ln C + 2 \ln K + (r - 2C)K\tilde{V}^{pa}_K(K)\}}_{\equiv H^{pa}(K)}. \tag{7}$$

We note that this is the HJB pertaining to a standard savings problem for a family with two individuals. $\tilde{V}^{pa}$ must be continuous and piecewise C^1 since $\tilde{V}$ is piecewise smooth. We now show that there cannot be kinks in $\tilde{V}^{pa}$ either. Consider two sequences $K + \frac{1}{n}$ and $K - \frac{1}{n}$. $\lim_{n\to\infty} \tilde{V}^{pa}(K - \frac{1}{n}) = \lim_{n\to\infty} \tilde{V}^{pa}(K + \frac{1}{n})$ on the left-hand side of (7) implies $\lim_{n\to\infty} H^{pa}(K - \frac{1}{n}) = \lim_{n\to\infty} H^{pa}(K + \frac{1}{n})$ on the right-hand side. Since H^{pa} is strictly decreasing and continuous, it follows that $\lim_{n\to\infty} \tilde{V}^{pa}_K(K - \frac{1}{n}) = \lim_{n\to\infty} \tilde{V}^{pa}_K(K + \frac{1}{n})$ and thus there cannot be a kink at K. From the standard consumption-savings (or Merton) problem we know that the unique smooth solution to (7) is the value function associated with the Pareto problem in (11) in the paper, where $\alpha = \alpha' = 1$. Agents' consumption strategies thus must be $C = C' = \rho/2$ and transfers are as claimed in point 2a. ■

5 NT-regions: dynamics at steady state

This section will study the Euler equations inside NT-regions for the special case where $c = c'$ and thus $\dot{P} = 0$, i.e. at stationary points. We first treat the general case where the consumption rates do not equal the SS consumption rates; Subsection 5.1 then studies the special case where $c = c' = \rho$ (i.e. both players consume at the SS rates). The latter case is especially difficult because the Euler equations have a singularity at this point.

We first bring the Euler equations (equation (1) for him and equation (29) in the paper for her) for NT-regions in (P, K)-space in a convenient form, using both C, C' and c, c'. In a no-transfer region with $G = G' = 0$, when using the FOC (equation (28) in the paper for her) for consumption, we can obtain the following Euler equations in consumption rates (C, C') out of the total wealth K:

$$-\dot{P}\frac{C_P}{C^2} = \left[\frac{1}{C} - \frac{1+\alpha}{\rho}\right]\left\{(\rho - C - C') - \frac{\dot{P}}{(1-P)}\right\} + \left[\left(\frac{1+\alpha}{\rho} - \frac{P}{C}\right) - \alpha\frac{(1-P)}{C'}\right] C'_P,$$

$$\dot{P}\frac{C'_P}{C'^2} = -\left[\frac{1}{C'} - \frac{1+\alpha'}{\rho}\right]\left\{(\rho - C - C') + \frac{\dot{P}}{P}\right\} + \left[\left(\frac{1+\alpha'}{\rho} - \frac{(1-P)}{C'}\right) - \alpha'\frac{P}{C}\right] C_P.$$

It is then instructive to replace some of the terms in (C, C') in the above equations by consumption rates (c, c') out of agents' own assets (k, k'), for which we recall that $C = cP$ and $C' = c'(1-P)$:

$$-\underbrace{\left[\frac{1-P}{C}\left(\frac{c'-c}{c}\right)\right]}_{\equiv q_1(P,c,c')} C_P = \underbrace{\left[\frac{1}{C} - \frac{1+\alpha}{\rho}\right][\rho - c']}_{\equiv q_3(P,c,c')} + \underbrace{\left[\left(\frac{1}{\rho} - \frac{1}{c}\right) + \alpha\left(\frac{1}{\rho} - \frac{1}{c'}\right)\right]}_{\equiv q_2(c,c')} C'_P, \tag{8}$$

$$-\underbrace{\left[\frac{P}{C'}\left(\frac{c-c'}{c'}\right)\right]}_{\equiv q'_1(P,c,c')} C'_P = -\underbrace{\left[\frac{1}{C'} - \frac{1+\alpha'}{\rho}\right][\rho - c]}_{\equiv q'_3(P,c,c')} + \underbrace{\left[\left(\frac{1}{\rho} - \frac{1}{c'}\right) + \alpha'\left(\frac{1}{\rho} - \frac{1}{c}\right)\right]}_{\equiv q'_2(c,c')} C_P, \tag{9}$$

where we now see that the cases $c = c'$, $c = \rho$ and $c' = \rho$ are special. Indeed, if $c = c' = \rho$, all terms vanish and there is a singularity if we approach the SS rates (see the discussion in Subsection 5.1).

For the case $c = c' = c_0 \neq \rho$, we have $q_1 = q'_1 = 0$ and thus

$$C_P = -\frac{\frac{1}{C'} - \frac{1+\alpha'}{\rho}}{1+\alpha'}\left(\frac{c_0 - \rho}{\frac{1}{\rho} - \frac{1}{c_0}}\right) \leq 0, \qquad C'_P = \frac{\frac{1}{C} - \frac{1+\alpha}{\rho}}{1+\alpha}\left(\frac{c_0 - \rho}{\frac{1}{\rho} - \frac{1}{c_0}}\right) \geq 0.$$

The inequalities follow from the parentheses always being positive and the fact that $C \leq \rho/(1+\alpha)$ and $C' \leq \rho/(1+\alpha')$, which in turn follows from $V_P \geq 0$ (see Proposition 2 in the main paper) and the FOC (equation (28) in the main paper). From the identities $c = CP$ and $c' = C'(1-P)$ it then follows that

$$c_P = \frac{C_P - c}{P} < 0, \qquad c'_P = \frac{C'_P + c'}{1-P} > 0.$$

So it must be that $\dot{P} > 0$ immediately to the right of a stationary point P^* and $\dot{P} < 0$ immediately to the left of P^*. Thus P^* is an unstable steady state, as claimed in the paper.

5.1 Singularity of Euler equation at SS-NT boundary

In the following subsection, we show that the dynamics of the economy are the same in the neighborhood of an SS-region as in the neighborhood of a "regular" steady state with $c = c' \neq \rho$: the economy moves away from the SS-region. To do this, we analyze the system of Euler equations in the neighborhood of the SS consumption rates $c = c' = \rho$. We first note that value-matching implies that the limit consumption rates in the NT-region must be the SS ones, since this is the only solution to the system of HJBs in this case (see case 4 in Section 6.5).

For convenience, we reproduce the Euler equations entirely in "small" consumption rates, using the above-defined functions $\{q_i\}$:

$$\begin{aligned} q_1 P c_P + q_2(1-P)c'_P &= -q_1 c + q_2 c' - q_3, \\ q'_2 P c_P + q'_1(1-P)c'_P &= q'_1 c - q'_2 c' + q'_3. \end{aligned} \tag{10}$$

In the case $c = c' = \rho$, the Euler equations per se cannot tell us anything about the derivatives c_P and c'_P since $q_1 = q_2 = q_3 = q'_1 = q'_2 = q'_3 = 0$ in (10) – any pair (c_P, c'_P) is a solution to the above system for given (P, c, c'). However, in a neighborhood around the rates (ρ, ρ) the equations still contain information since the q-functions have not yet vanished. We will study solutions to the system of Euler equations in the neighborhood of $(c, c', P) = (\rho, \rho, P)$ by linearizing the Euler equations around this point. This technique is equivalent to applying L'Hospital's Rule in multiple dimensions – we don't divide zero by zero but study derivatives instead.

We first calculate the partial derivatives of $(q_i, q'_i)_{i=1}^3$ in (c, c') at $(c, c') = (\rho, \rho)$ for a given P – note that all derivatives in P vanish since $c = c' = \rho$ and thus $q_1 = q'_1 = q_2 = q'_2 = 0$:

$$\frac{\partial q_1}{\partial c} = -\frac{1-P}{P\rho^2}, \quad \frac{\partial q_1}{\partial c'} = \frac{1-P}{P\rho^2}; \quad \frac{\partial q'_1}{\partial c} = \frac{P}{(1-P)\rho^2}, \quad \frac{\partial q'_1}{\partial c'} = -\frac{P}{(1-P)\rho^2};$$

$$\frac{\partial q_2}{\partial c} = \frac{1}{\rho^2}, \quad \frac{\partial q_2}{\partial c'} = \frac{\alpha}{\rho^2}; \quad \frac{\partial q'_2}{\partial c} = \frac{\alpha'}{\rho^2}, \quad \frac{\partial q'_2}{\partial c'} = \frac{1}{\rho^2};$$

$$\frac{\partial q_3}{\partial c} = 0, \quad \frac{\partial q_3}{\partial c'} = \frac{(1+\alpha) - \frac{1}{P}}{\rho}; \quad \frac{\partial q'_3}{\partial c} = \frac{(1+\alpha') - \frac{1}{1-P}}{\rho}, \quad \frac{\partial q'_3}{\partial c'} = 0.$$

We then use these derivatives to linearize the system of ODEs around (ρ, ρ, P), where P is arbitrary – note that all terms in ΔP vanish. In matrix from, we have for a small deviation $(\Delta c, \Delta c', \Delta P)$ from (ρ, ρ, P) to a first order:

$$\left[\Delta c \begin{pmatrix} \frac{\partial q_1}{\partial c} P & \frac{\partial q_2}{\partial c}(1-P) \\ \frac{\partial q_2'}{\partial c} P & \frac{\partial q_1'}{\partial c}(1-P) \end{pmatrix} + \Delta c' \begin{pmatrix} \frac{\partial q_1}{\partial c'} P & \frac{\partial q_2}{\partial c'}(1-P) \\ \frac{\partial q_2'}{\partial c'} P & \frac{\partial q_1'}{\partial c'}(1-P) \end{pmatrix}\right] \begin{pmatrix} c_P \\ c_P' \end{pmatrix} =$$

$$= \Delta c \begin{pmatrix} -\frac{\partial q_1}{\partial c}\rho + \frac{\partial q_2}{\partial c}\rho - \frac{\partial q_3}{\partial c} \\ \frac{\partial q_1'}{\partial c}\rho - \frac{\partial q_2'}{\partial c}\rho + \frac{\partial q_3'}{\partial c} \end{pmatrix} + \Delta c' \begin{pmatrix} -\frac{\partial q_1}{\partial c'}\rho + \frac{\partial q_2}{\partial c'}\rho - \frac{\partial q_3}{\partial c'} \\ \frac{\partial q_1'}{\partial c'}\rho - \frac{\partial q_2'}{\partial c'}\rho + \frac{\partial q_3'}{\partial c'} \end{pmatrix}.$$

Note that multiplying Δc and $\Delta c'$ by the same constant leaves us with the same solutions. This is to be expected when we linearize: the solutions are the same on any ray going away from (ρ, ρ) in the (c, c')-space, only the angle matters. We thus set $\Delta c = \sin\phi$ and $\Delta c' = \cos\phi$ for $\phi \in [0, 2\pi)$ to study all rays leading away from (ρ, ρ). Simplifications yield

$$\frac{1}{\rho}\underbrace{\left[\sin\phi \begin{pmatrix} P-1 & 1-P \\ \alpha' P & P \end{pmatrix} + \cos\phi \begin{pmatrix} 1-P & (1-P)\alpha \\ P & -P \end{pmatrix}\right]}_{\equiv A(\phi)} \begin{pmatrix} c_P \\ c_P' \end{pmatrix} =$$

$$= \sin\phi \begin{pmatrix} \frac{1}{P} \\ 0 \end{pmatrix} + \cos\phi \begin{pmatrix} 0 \\ -\frac{1}{1-P} \end{pmatrix}.$$

First, we see that the system will have the same solutions for for $\phi + \pi$ and ϕ, since $\sin(\phi+\pi) = -\sin\phi$ and $\cos(\phi+\pi) = -\cos(\pi)$. Multiplying the system by -1 (or going into the exactly opposite direction in (c, c')-space) yields of course the same solutions, so we may restrict our analysis to the half-circle $(-\pi/2, \pi/2]$. See Figure 1 to gain some intuition.

If $A(\phi)$ is invertible (i.e. $|A(\phi)| \neq 0$), then the unique solution is given by

$$\begin{pmatrix} c_P \\ c_P' \end{pmatrix}(\phi) = \frac{1}{A(\phi)} \begin{pmatrix} \sin^2\phi + \alpha\cos^2\phi \\ -\alpha'\sin^2\phi - \cos^2\phi \end{pmatrix}.$$

We see that c_P and c_P' must always be of opposite sign, so arrows in phase space must point north-west, as is evident in the figure. This also immediately implies that there cannot be any path coming from or going into the north-east quadrant, so we can restrict our analysis to the north-west quadrant, which means considering only the interval $\phi \in [-\pi/2, 0]$.

The following argument shows that indeed $|A(\phi)| < 0$ on this range, so there is a unique solution to the system. Consider the determinant of $A(\phi)$:

$$\det A(\phi) = -P(1-P)\left[(1+\alpha')\sin^2\phi + (1+\alpha)\cos^2\phi - (1-\alpha\alpha')\cos\phi\sin\phi\right].$$

The following shows that $A(\phi) < 0$ for all ϕ:

$$0 < \left(\sqrt{1+\alpha'}\sin\phi - \sqrt{1+\alpha}\cos\phi\right)^2 =$$

$$=(1+\alpha')\sin^2\phi + (1+\alpha)\cos^2\phi - 2\sqrt{1+\alpha'}\sqrt{1+\alpha}\sin\phi\cos\phi \le -\frac{A(\phi)}{P(1-P)},$$

where the last step follows from $2\sqrt{1+\alpha'}\sqrt{1+\alpha} > 1 \geq 1 - \alpha\alpha'$ for any (α, α'). So we conclude that the above system has a unique solution for all ϕ.

We conclude that $c_P(\phi) < 0$ and $c'_P(\phi) > 0$ for all directions ϕ. Note that since $\sin(\phi + \pi) = -\sin\pi$ and $\cos(\phi+\pi) = -\cos(\phi)$, $A(\phi) = A(\phi+\pi)$ and $\mathbf{c}_P(\phi + \pi) = \mathbf{c}_P(\pi)$. So completing half a circle leads to the same vector direction, as is evident in Figure 1.

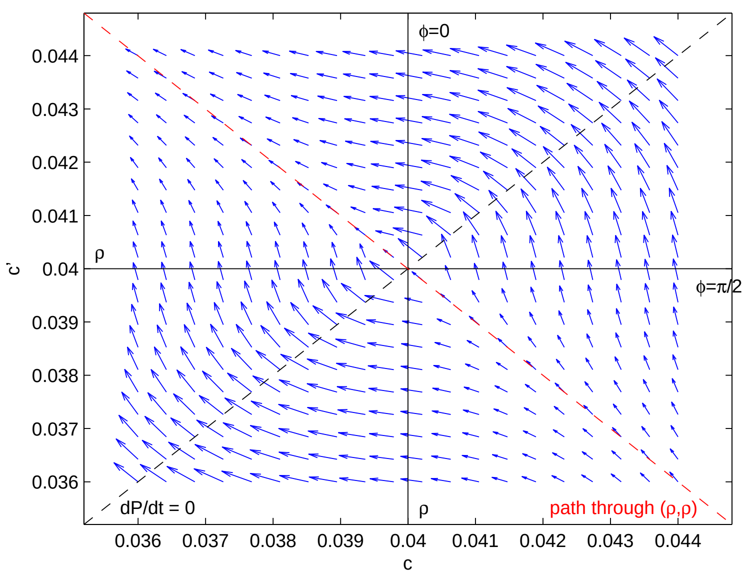

Figure 1: $\mathbf{c}_P$ around (ρ, ρ) (for $\alpha = \alpha' = 0.2$)

We are looking for the angle $\phi \in [0, \pi)$ of the path leading through (ρ, ρ). Since $\sin^2\phi + \cos^2\phi = 1$, the angle of the vector $\mathbf{c}_P = (c_P, c'_P)$ is

$$\tan\xi(\phi) = -\frac{(1-\alpha)\sin^2\phi + \alpha}{1 - (1-\alpha')\sin^2\phi} \equiv -\frac{N(\phi)}{D(\phi)}.$$

It is easy to see that $N'(\phi)$ has the opposite sign of $D'(\phi)$; $\tan\xi$ is maximized at $\phi = 0$ with value $-\alpha$, it is decreasing on $[0, \pi/2]$, takes its minimum at $\phi = \pi/2$ with value $-1/\alpha'$ and increases again on $[\pi/2, \pi)$.

A path $\bar{c}(P)$ going through (ρ, ρ) must be such that $\xi(\phi) = \phi$, or equivalently – since $\tan(\cdot)$ is a strictly increasing function on $[-\pi/2, \pi/2)$–:

$$\tan\xi(\phi) = \tan\phi.$$

Since $\tan\xi(\phi) \leq 0$ for all ϕ and $\tan(\phi) > 0$ for $\phi > 0$, there cannot be any solution to this fixed-point problem on the range $(0, \phi/2)$. On $[-\pi/2, 0]$, there must exist at least one fixed point since $\tan(\phi)$ crosses the entire range of $\tan\xi(\phi)$.

So we conclude that it is possible that a SS-region borders an NT-region, but that consumption rates (c, c') must converge to (ρ, ρ) from a certain angle. From the phase diagram, we see that c' is increasing in P and c is decreasing in P, so $\dot{P} < 0$ if NT is to the left of SS and $\dot{P} > 0$ if NT is to the right of NT. So if a SS-region was contained as a steady state within a NT-region, we have the same dynamics as with other steady states within NT-regions: the economy would always move away from SS.

6 Restrictions from value matching (VM)

In order to construct (or rule out) equilibria with more than one region, it is essential to characterize the conditions imposed by value-matching (VM) on policies left and right of a boundary. This is especially challenging for NT-regions; we provide a discussion of such boundaries in this section.

Consider two regions A and B, where $A = \mathcal{P}_i$ lies to the left of $B = \mathcal{P}_{i+1}$. We denote the boundary as $\tilde{P} = P_i$. It turns out that it is convenient to work with policies in terms of agents' own wealth; we denote $c_A = \lim_{P \to \tilde{P}^-} c(P)$ as the limit of consumption on the A-side. The notation for c'_A, c_B, g_A and so forth is analogous. The policies directly on the boundary are denoted by $c_K = c(\tilde{P})$ etc., where K stands for "kink". We write H_A for the Hamiltonian using V_p^- and analogously H_B for the Hamiltonian using V_p^+.

The following proposition summarizes our results on the case where the value functions are given on the A-side and we are looking for the consumption policies on the B-side:

Proposition 2 (Discontinuities in policies) *Consider a region A that lies to the left of region B, separated by a boundary $\tilde{P}$. Suppose that B is a NT-region, and let values V_A and V'_A be given on the A-side. Then, in any homogeneous MPE, we have:*

1. *If the consumption policies are $c_A = c'_A = \rho$ on the A-side, then also $c_B = c'_B = \rho$ on the B-side.*

2. *If either $c_A \neq \rho$ or $c'_A \neq \rho$, then there are at least two candidates for the consumption policies $\{c_B, c'_B\}$ consistent with value matching. One of these candidates coincides with $\{c_A, c'_A\}$. Furthermore:*

   (a) *If also region A is of NT-type, then*

$$\begin{aligned} c_A > c'_A \quad &\Leftrightarrow \quad c_B < c'_B, \\ c_A < c'_A \quad &\Leftrightarrow \quad c_B > c'_B, \\ c_A = c'_A \quad &\Leftrightarrow \quad c_B = c'_B. \end{aligned}$$

   (b) *Let $k(\cdot, \cdot)$ and $k'(\cdot, \cdot)$ be given by (11) and (12). If either $k(c_A, c'_A) \leq k(\rho, \rho)$ or $k'(c_A, c'_A) \leq k(\rho, \rho)$, then there are exactly two solutions for*

(c_B, c_B'). *These solutions have the following property: If he is under-consuming in her eyes on the A-side, he must be over-consuming in her eyes on the B-side:*

$$\frac{\alpha}{c_A'} > \frac{1+\alpha}{\rho} - \frac{1}{c_A} \qquad \Leftrightarrow \qquad \frac{\alpha}{c_B'} > \frac{1+\alpha}{\rho} - \frac{1}{c_B}.$$

The same is true reversing the inequalities, and for her under-/over-consumption in a symmetric fashion.

Before we start to prove the proposition, we establish the following corollary, which is what we will need to rule out the NT-NT-NT structure:

Corollary 1 (No attracting boundary between NT-regions) *Consider regions A, B separated by a boundary $\tilde{P}$ as in Proposition 2. If both A and B are NT, then $\dot{P}_A > 0$ and $\dot{P}_B < 0$ (at the same time) cannot occur in any homogeneous MPE.*

Proof: Since both A and B are NT, transfer motives are strictly negative at $\tilde{P}$ in both directions. Thus transfers by both players are zero on the kink and $\dot{P}_K$ is solely determined by c_K and c_K'. We will now show that it is impossible that $\dot{P}_K < 0$. This would imply $(c_K, c_K') = (c_A, c_A')$ by Lemma 4 in the paper, which in turn implies $\dot{P}_K = \dot{P}_B > 0$, a contradiction. In the same way, $\dot{P}_K > 0$ is ruled out. Finally, $\dot{P}_K = 0$ is impossible since there are no consumption policies $c_K = c_K'$ that are consistent with value matching by point 2a of Proposition 2. This leads us to conclude that no such boundary exist in equilibrium. ■

Repelling boundaries (i.e. $\dot{P}_A < 0$ and $\dot{P}_B > 0$) are possible if they satisfy value matching. The policies on the two sides are characterized by Proposition 2. Both the A- and the B-side policies are then possible policies on the boundary.

We now proceed to prove Proposition 2. The proof is constructive and provides an algorithm for finding the second solution to the system of VM conditions. In one specific case we were not able to prove that there are *exactly* two solutions. Numerical exercises, however, suggest that also in this case there exists no third solution.

To start, we first state her HJB (equation 20 in the paper) replacing V_P in terms of C using the FOC (equation 22 in the paper) and doing the same for him:

$$\rho V = \alpha \ln C' - \frac{C'}{1-P}\left[\frac{1+\alpha}{\rho} - \frac{P}{C}\right] \ln C - 1,$$
$$\rho V' = \alpha' \ln C - \frac{C}{P}\left[\frac{1+\alpha'}{\rho} - \frac{1-P}{C'}\right] + \ln C' - 1.$$

Now, expressing consumption rates in terms of the agents' own wealth, i.e. using the identities $c = C/P$ and $c' = C'/(1-P)$), we obtain

$$\rho V = \ln P + \alpha \ln(1-P) + \underbrace{\alpha \ln c' - c' \left[\frac{1+\alpha}{\rho} - \frac{1}{c}\right]}_{\equiv g(c,c')} + \underbrace{\ln c - 1}_{\equiv h^*(c)},$$

$$\rho V' = \ln(1-P) + \alpha' \ln P + \underbrace{\alpha' \ln c - c \left[\frac{1+\alpha'}{\rho} - \frac{1}{c'}\right]}_{\equiv g'(c,c')} + \underbrace{\ln c' - 1}_{\equiv h^*(c')}.$$

We see that two regimes (c_A, c'_A) and (c_B, c'_B) can only be consistent with value matching if the following two VM conditions hold:

$$k(c_A, c'_A) \equiv g(c_A, c'_A) + h^*(c_A) = g(c_B, c'_B) + h^*(c_B) = k(c_B, c'_B), \tag{11}$$

$$k'(c_A, c'_A) \equiv g'(c_A, c'_A) + h^*(c'_A) = g'(c_B, c'_B) + h^*(c'_B) = k'(c_B, c'_B). \tag{12}$$

We see that this VM condition is independent of $\tilde{P}$, which will facilitate our analysis.

We now want to determine what the relevant range for consumption policies is. The FOCs for consumption together with $V_P \geq 0$ and non-negativity of consumption give us the following bounds:

$$0 \leq C \leq \frac{\rho}{1+\alpha}, \qquad 0 \leq c \leq \frac{\rho}{\tilde{P}(1+\alpha)}; \tag{13}$$

$$0 \leq C' \leq \frac{\rho}{1+\alpha'}, \qquad 0 \leq c \leq \frac{\rho}{(1-\tilde{P})(1+\alpha')}. \tag{14}$$

Note that these bounds *do* depend on $\tilde{P}$, so we cannot neglect $\tilde{P}$ altogether when trying to determine (c_B, c'_B) for a given pair (c_A, c'_A).

In order to characterize the solution, it will be crucial to study the derivatives of $k(\cdot,\cdot)$. We start with the derivative in an agent's own consumption:

$$\frac{\partial k(c,c')}{\partial c} = \frac{1}{c}\left(1 - \frac{c'}{c}\right), \qquad \frac{\partial k'(c,c')}{\partial c'} = \frac{1}{c'}\left(1 - \frac{c}{c'}\right). \tag{15}$$

Fixing c', k is decreasing in c for $c < c'$ (i.e. above the diagonal of the (c,c')-plane) and increasing in c for $c > c'$ (below the diagonal). For given c, k is minimized by $c' = c$.

The derivative in the other agent's consumption is

$$\frac{\partial k(c,c')}{\partial c'} = \frac{\alpha}{c'} - \left[\frac{1+\alpha}{\rho} - \frac{1}{c}\right], \qquad \frac{\partial k'(c,c')}{\partial c} = \frac{\alpha'}{c} - \left[\frac{1+\alpha'}{\rho} - \frac{1}{c'}\right]. \tag{16}$$

Note that if the bracket on the right-hand side of (16) is negative ($c \leq \rho/(1+\alpha)$), then an increase in c' always leads to an increase in k. In this case, her marginal value V_P of having the wealth distribution tilted in her favor is so high that this dominates the marginal value of common funds $(1+\alpha)/\rho$; she would set infinite consumption for him if she had the choice. If the bracket is positive, $\partial k/\partial c'$ is strictly decreasing in c' for fixed c.

We now introduce the following related maximization problem:

$$\begin{aligned} k^*(c) &\equiv \max_{c'} k(c,c'), \\ c'^*(c) &\equiv \arg\max_{c'} k(c,c') = \frac{\alpha}{\max\left\{\frac{1+\alpha}{\rho} - \frac{1}{c}, 0\right\}}. \end{aligned} \tag{17}$$

Since this is a strictly concave problem, the FOC (16) is sufficient. $c'^*(c)$ has the interpretation as the consumption rate that she would choose for him given c. Note that we have $c'^* = \alpha/0 = \infty$ in the case that $1/c$ is large. In (c,c')-space, k is increasing in c' below the function $c'^*(c)$ and decreasing in c' above, as the FOC (16) shows.

6.1 Properties of $c'^*(\cdot)$ and $c^*(\cdot)$

Our goal will now be to show that (i) the unique intersection between the functions $c'^*(\cdot)$ and $c^*(\cdot)$ in the (c,c')-plane is at the point (ρ,ρ), (ii) the graph of c'^* lies *above* the graph of c^* for values $c > \rho$ and (iii) c'^* lies *below* c^* for values $c < \rho$.

We first re-write $c'^*(\cdot)$ in the area where it is bounded:

$$c'^*(c) = \frac{1}{\frac{1}{\rho} + \frac{1}{\alpha}\left(\frac{1}{\rho} - \frac{1}{c}\right)}.$$

Now, we invert this function and then reverse the roles of the two players to obtain the graph of c^* in the (c,c')-plane as a function c' of c. We call the inverted function $\tilde{c}' \equiv (c^*)^{-1}$:

$$\tilde{c}'(c) = \frac{1}{\frac{1}{\rho} + \alpha'\left(\frac{1}{\rho} - \frac{1}{c}\right)}.$$

Inspection of the two functions makes clear that the claims (i)-(iii) are true whenever $\alpha\alpha' < 1$, see also Figure 2 for an illustration. The functions fall on top of each other in the case $\alpha = \alpha' = 1$.

6.2 Properties of $k^*(c)$ and $k'^*(c')$

By substituting $c'^*(c)$ into $k(c,c')$, it can be verified that $k^*(\cdot)$ has the following properties:[9]

1. k^* is smooth and strictly convex.

[9] It is worthwhile noting that these properties do not depend on the choice $u(c) = \ln c$ for instantaneous utility. They can be derived generally for concave utility functions $u(\cdot)$ satisfying $\lim_{c\to 0} u(c) = -\infty$ and $\lim_{c\to\infty} u(c) = \infty$ using only the convexity properties of Legendre transforms.

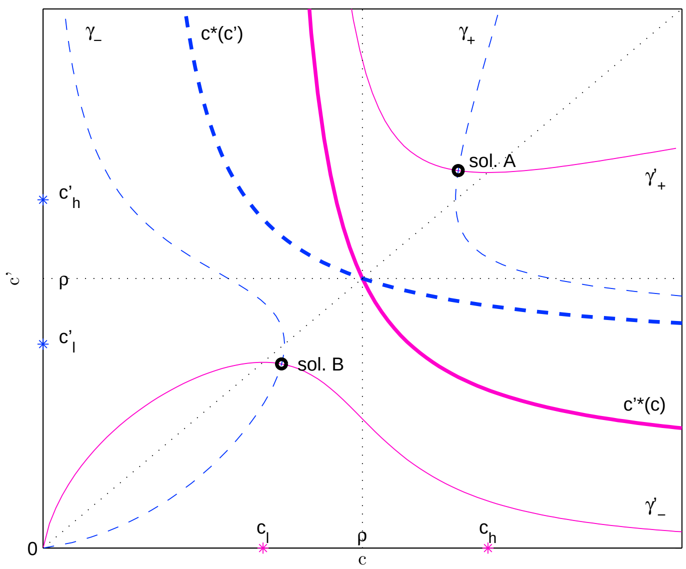

Figure 2: $k_A < k_\rho$ and $k'_A < k'_\rho$ (case 1)

2. k^* is uniquely minimized at $c = \rho$.
3. $\lim_{c \to 0} k^*(c) = \lim_{c \to \infty} k^*(c) = \infty$.

The same is true for the function $k'^*(\cdot)$, making the obvious adjustments. The above properties together with the derivatives (15) and (16) imply that the point (ρ, ρ) is the saddle point of both functions $k^*(\cdot)$ and $k'^*(\cdot)$: Geometrically, $c'^*(c)$ is the ridge of a mountain (the maximal point of the k-surface keeping c fixed and varying c'). At $c = \rho$, this ridge attains its lowest point. The equivalent statements hold for $k'^*(\cdot)$ and $c^*(\cdot)$.

6.3 Find c' given c to solve her VM

We will now study the following problem: given a fixed c, find c' to solve $k(c, c') = k_A$ (her VM). In order to find the solution(s), it will be useful to study the function $g^c(c') \equiv g(c, c')$. Obviously, $g^c(\cdot)$ is smooth and concave, and we have $\lim_{c' \to 0} g^c(c') = -\infty$.

There are two cases to consider:

1. $c > \rho/(1 + \alpha)$: $g^c(\cdot)$ attains its unique maximum at $c'^*(c)$ and we have $\lim_{c' \to \infty} g^c(c') = -\infty$. There are the following sub-cases:

(a) $k^*(c) > k(c_A, c'_A)$: There are two solutions; we denote the upper solution by $\gamma'_+(c)$ and the lower one by $\gamma'_-(c)$. Of course, $\gamma'_+(c) > c'^*(c) > \gamma'_-(c)$.

(b) $k^*(c) = k(c_A, c'_A)$: There is exactly one solution $\gamma'(c) = c'^*(c)$.

(c) $k^*(c) < k(c_A, c'_A)$: There is no solution.

This case corresponds to the area to the right of c_l in Figure 2 where $c'^* < \infty$.

2. $c \leq \rho/(1+\alpha)$: g^c is globally increasing and $\lim_{c' \to \infty} g^c(c') = \infty$. Thus there is exactly one value for c' that solves VM, which we denote by $\gamma'_-(c)$, since it lies below $c'^*(c) = \infty$. This case corresponds to the area to the left of c_l in Figure 2 where $c'^* = \infty$.

The functions γ'_+ and γ'_- are drawn as thin solid lines in Figure 2. They correspond to the locus where her VM is fulfilled. When drawing the analogous lines γ_+ and γ_- for him (the dotted thin lines), the intersections of the two give us the points where VM is fulfilled for both players, which are the solutions that we are ultimately after. In the following, we will be concerned with characterizing the γ-functions in order to be able to characterize these solutions.

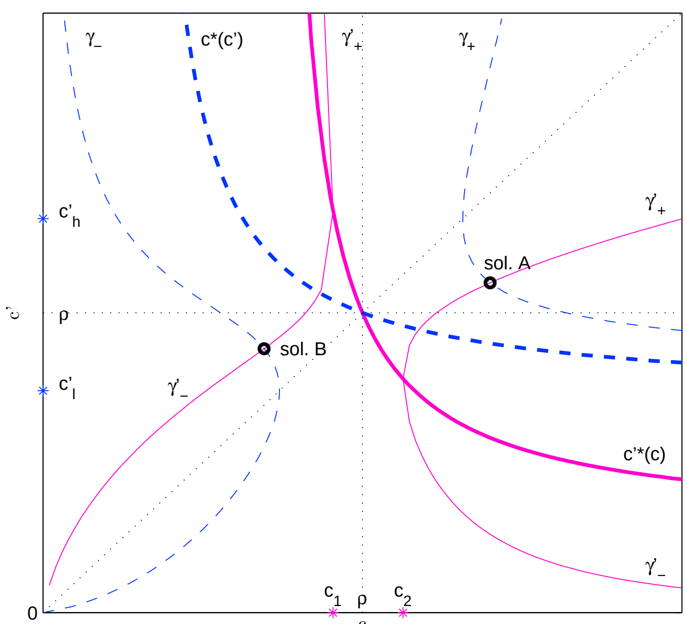

Figure 3: $k_A \leq k_\rho$ and $k'_A \geq k'_\rho$ (case 2)

6.4 Characterization of γ-functions

Since $k(\cdot,\cdot)$ is a smooth function, both $\gamma'_-(\cdot)$ and $\gamma'_+(\cdot)$ will be smooth functions (on the range where they are defined). The implicit-function theorem gives us

$$\frac{\partial\gamma'}{\partial c} = \frac{dc'}{dc} = -\frac{\frac{1}{c}\left(1-\frac{c'}{c}\right)}{\frac{\alpha}{c'}-\left(\frac{1+\alpha}{\rho}-\frac{1}{c}\right)} = \frac{(c'-c)c}{cc'+\alpha c^2-\frac{1+\alpha}{\rho}cc'} \tag{18}$$

$$\frac{\partial\gamma}{\partial c'} = \frac{dc}{dc'} = -\frac{\frac{1}{c'}\left(1-\frac{c}{c'}\right)}{\frac{\alpha'}{c}-\left(\frac{1+\alpha'}{\rho}-\frac{1}{c'}\right)} = \frac{(c-c')c'}{c'c+\alpha' c'^2-\frac{1+\alpha'}{\rho}c'c}, \tag{19}$$

where of course we have to set $c' = \gamma'(c)$ along the graph of $\gamma'(\cdot)$ and $c = \gamma(c')$ along the graph of $\gamma(\cdot)$. The denominator $\partial g(c,c')/\partial c'$ is equal to the bracket in the altruistic-strategic distortion of her Euler equation (see equation 15 in the paper). Along $\gamma'_+(\cdot)$, $\partial g(c,c')/\partial c'$ is always negative, which means that she would prefer him to consume less (*over-consumption*). Along γ'_-, $\partial g(c,c')/\partial c'$ is always positive, which means that she would prefer him to consume more (*under-consumption*).

Furthermore, we see that the numerator in (18) is zero if and only if $c = c'$, i.e. on the diagonal of the (c,c')-plane; it is positive above the diagonal and negative below. The denominator coincides with the FOC of the problem (17), so it is zero on $c'^*(c)$, positive below and negative above. So γ'_+ (which lies above c'^*) is decreasing above the diagonal and increasing below it. γ'_- (which lies below c'^*) is increasing above the diagonal and decreasing below. Furthermore, γ'_+ and γ'_- have infinite slope at the point where they cross the function $c'^*(c)$, which is the point where the two solutions collapse to a single one; see the right pink curve in Figure 4 for an illustration.

Since the diagonal $c = c'$ plays a crucial role, it is useful to study the function $k(\cdot,\cdot)$ along it. Define

$$k_{symm}(c) \equiv k(c,c) = (1+\alpha)\left(\ln c - \frac{c}{\rho}\right).$$

It is clear that $k_{symm}(\cdot)$ is smooth, convex, uniquely maximized at $c = \rho$ and that $\lim_{c\to 0} k_{symm}(c) = \lim_{c\to\infty} k_{symm}(c) = -\infty$.

We will now characterize the γ-functions. It turns out that the value of $k(\cdot)$ on the saddle point $k_\rho \equiv k_{symm}(\rho) = k(\rho,\rho) = (1+\alpha)(\ln\rho - 1)$ plays a crucial role in distinguishing different cases:

1. $k_A < k_\rho$: for each $c \in (0,\infty)$, $\gamma'_+(c)$ and $\gamma'_-(c)$ exist since $k^*(c) > k_\rho$ for all c. By the properties of $k_{symm}(\cdot)$, there are exactly two numbers c_l (with $0 < c_l < \rho$) and c_h (with $c_h > \rho$) such that $k_{symm} = k_A$. This implies that γ'_- is a smooth function on $(0,\infty)$, is uniquely maximized at c_l, is increasing and above the diagonal for $c < c_l$ and decreasing and below the diagonal for $c > c_l$. γ'_+ is a smooth function on $(\rho/(1+\alpha),\infty)$, is uniquely minimized at c_h, is decreasing above the diagonal and increasing below it. The situation is illustrated in Figure 2 by the solid thin lines.

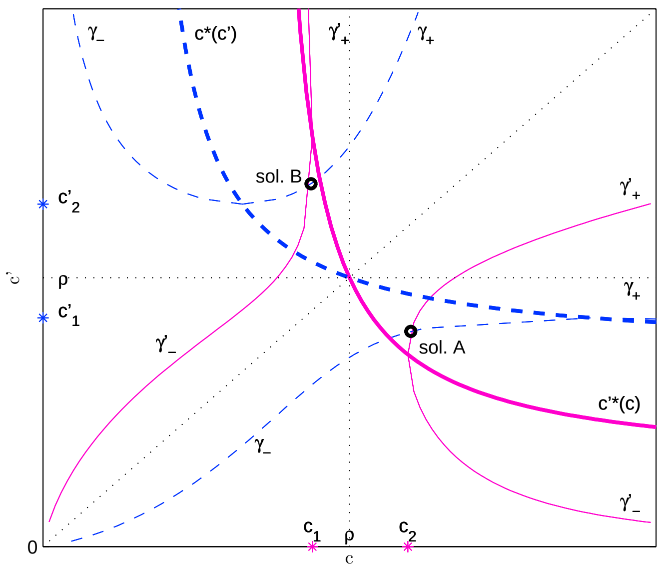

Figure 4: $k_A \geq k_\rho$ and $k'_A \geq k'_\rho$ (case 3)

2. $k_A > k_\rho$: By the properties of $k^*(\cdot)$, there are exactly two values $c_1 < \rho$ and $c_2 > \rho$ such that $k^*(c_1) = k^*(c_2) = k_A$. No solution c' exist for $k(c, c') = k_A$ for given $c \in (c_1, c_2)$ since $k^*(c) < k_A$, so the γ-functions are not defined on this range. Also note that the γ-functions cannot cross the diagonal since $k_{symm}(c) \leq k_\rho < k_A$.Thus γ'_- is an increasing function that stays above the diagonal for $c \in (0, c_1)$ and a decreasing function that stays below the diagonal for $c \in (c_2, \infty)$. γ'_+ is a decreasing function above the diagonal for $c \in (0, c_1)$ and an increasing function below the diagonal for $c \in (c_2, \infty)$. The situation is illustrated in Figure 4.

3. $k_A = k_\rho$: By the same argument as in the case $k_A < k_\rho$, γ'_+ and γ'_- are defined for all values $c \in (0, \infty)$, if we allow the coincidence $\gamma'_+(\rho) = \gamma'_-(\rho) = \rho$ on the saddle point (ρ, ρ). Since $k(\cdot, \cdot)$ is differentiable, level lines must be differentiable, too. So γ'_- must have the same slope just left of ρ as γ'_+ has just right of ρ. Also, it must be that γ'_- has slope smaller than unity just right of $c = \rho$: If γ'_- came out above the diagonal, (18) tells us that its slope should be negative, which is a contradiction. So $\rho < \gamma'_+(c) < c$ for all $c > \rho$. This in turn implies that γ'_- comes out above the diagonal just left of ρ and that $\gamma'_-(c) > c$ for all $c < \rho$ and $\rho < \gamma'_+(c) < c$. For the other two branches, we clearly have $\gamma'_+(c) > c'^*(c) > \rho$ for all $c < \rho$ and $\gamma'_-(c) < c'^*(c) < \rho$ for all $c > \rho$. The situation is illustrated in Figure 5.

Another important property, which is independent of the different cases, is the following: For c large enough, the function $\gamma'_+(c)$ always exists, stays below the diagonal and is increasing. As c goes to infinity, we have

$$\lim_{c\to\infty} \gamma'_+(c) = \infty, \qquad \lim_{c'\to\infty} \gamma_+(c') = \infty.$$

To see this, proceed by way of contradiction and suppose that there was a bound $\tilde{c}'$ to which γ'_+ converged. Then (18) would tell us that the slope of γ'_+ approaches unity as c grows large, which is a contradiction to γ'_+ being bounded.

6.5 Find (c, c') to solve both VM-conditions

We can now finally look at the different types of solutions to the system (11) and (12) of *both* VM-conditions. The following is an exhaustive list of the cases that arise. (Note that the arguments also provide algorithms to find the respective solutions.)

1. $k_A \leq k_\rho$ and $k'_A \leq k'_\rho$, where one of the inequalities is strict. Without loss of generality, assume that $c_l \leq c'_l$, i.e. γ'_- crosses the diagonal closer to the origin than γ_- does.[10] Then, there are exactly two solutions (see Figure 2 for an illustration):

   (a) Following γ'_- to the right from c_l on (i.e. on the range $c > c_l$), there must be a unique intersection point of γ'_- with γ_- (since γ'_- is decreasing in c, and γ_- is increasing in c' for $c' < c'_l \leq \rho$, see Subsection 6.4). This is a solution where both under-consume (we are on the γ_--parts), $c_B < \rho$, $c'_B < \rho$ and $c > c'$.

   (b) Notice that c_l is the lower solution to $\ln c - c/\rho - k_A/(1+\alpha) = 0$, and c'_l is the lower solution to $\ln c' - c'/\rho - k'_A/(1+\alpha') = 0$. It is easy to see that $c_l \leq c'_l$ implies $c_h \geq c'_h$ for the upper solutions. Now, proceed similarly as in 1a: Follow γ_+ upward from c'_h (i.e. on the range $c' \geq c'_h$). Since γ_+ is increasing in c' and γ'_+ is decreasing in c on $(\rho/(1+\alpha), c_h)$, there must be a unique intersection point (c_B, c'_B). This solution is such that both over-consume, $c_B > \rho$, $c'_B > \rho$ and $c < c'$. Note that on the other side of the diagonal ($c > c'$), the γ-functions cannot intersect due to the properties described in Subsection 6.4.

2. $k_A \leq k_\rho$ and $k'_A \geq k'_\rho$, where one of the inequalities is strict. Then, there are exactly two solutions. See Figure 3 for an illustration.

   (a) Again, follow γ'_- to the right starting at c_l (i.e. on the range $c > c_l$). Note that γ_- must cross $c'^*(\cdot)$ on its way to $c^*(\cdot)$ by the ordering of c^* and c'^* described in Subsection 6.1. Since γ'_- always stays below c'^*, there must be an intersection with γ_+. Again, by the properties of the γ-functions (see Subsection 6.4) this must be the only intersection

[10] Just reverse the roles of the two in case the crossing is the other way around.

that γ'_- can have with γ_- and γ_+. The solution has the property that both under-consume, $c_B \geq c'_B$ and $c'_B \leq \rho$.

(b) Now, consider γ'_+. By the properties of γ_+ and γ_-, it is clear that γ'_+ yields a solution to the right of the diagonal. When following γ'_+ to the left from c_h, γ'_+ grows unbounded as $c \to \rho/(1+\alpha)$. Since γ_+ grows unbounded in c', there must be a unique intersection of γ'_+ and γ_+. At this solution, both over-consume, $c \leq c'$ and $c' > \rho$.

3. $k_A \geq k_\rho$ and $k'_A \geq k'_\rho$, where one of the inequalities is strict: There are two sub-cases to consider, depending on the value of $k'(\cdot)$ at $(c_2, \gamma'(c_2))$[11]; see Figure 4 for an illustration (which corresponds to sub-case b). In sub-case a, we can show that there are exactly two solutions, in sub-case b there must be at least two (we could not rule out the possibility that there are more).

   (a) $k'(c_2, \gamma'(c_2)) \leq k'_A$: γ_- must lie below $(c_2, \gamma'(c_2))$ at $c = c_2$. When following γ'_- letting c increase from c_2 on, at least one intersection with γ_- must take place since γ_- must cut c^* at some point, and c^* is above c'^*, which again lies above γ'_-. This intersection must be unique, because k' strictly increases when we follow γ'_- south-east by the derivatives of k' (which are analogous to (15) and (16)). This solution is such that $c > c'$ and that both under-consume.

   (b) $k'(c_2, \gamma'(c_2)) > k'_A$: γ_- must lie above $(c_2, \gamma'(c_2))$ at $c = c_2$. When following γ'_+ letting c increase from c_2 on, at least one intersection with γ_- must take place since γ'_+ grows unbounded. This intersection must be such that $c > \rho > c'$, he over-consumes and she under-consumes. In this case, we cannot rule out that another crossing happens between γ_- and γ'_+, so there might be another solution. We could not find any such case computationally, though. Also, note that this is the *only* case in which there can be mixed solutions, i.e. where she over-consumes and he under-consumes.

   We can follow the same procedure starting at the point $(c'_2, c^*(c'_2))$ and will find (at least) one more solution there.

4. $k_A = k_\rho$ and $k'_A = k'_\rho$: All γ-functions (for both agents) must contain the point (ρ, ρ), which is one solution to the system. The properties of the γ-functions imply that there cannot be any other solution in the entire (c, c')-space, see Figure 5 for an illustration.

We now summarize and return to the big picture. We can always find one candidate solution (c_B, c'_B) for given (c_A, c'_A) that is different from (c_A, c'_A). This solution is independent of $\tilde{P}$. Of course we still have to check if it respects the bounds given in (13) and (14), which depend on P. In one sub-case, we could not rule out that there are even more solutions for (c_B, c'_B). But note that

[11] Recall that this is the locus where there is exactly one solution c' for her VM given c, so $\gamma'_-(c_2) = \gamma'_+(c_2) = c'^*(c_2)$. We thus simply write $\gamma'(c_2)$.

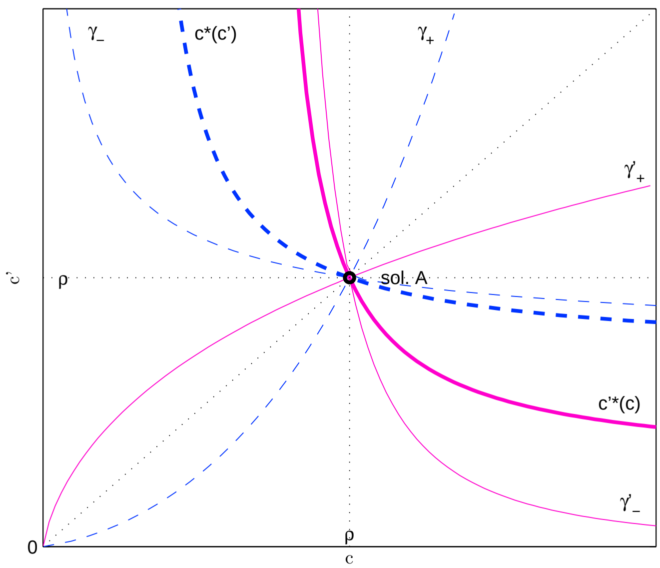

Figure 5: $k_A = k_\rho$ and $k'_A = k_\rho$ (case 4)

this does not weaken the point we make on the non-existence of NT-NT-NT equilibria, i.e. Corollary 1: Even for potential third solutions, the boundary $\tilde{P}$ would have to be attracting. Thus, there cannot exist policies on the kink $\tilde{P}$ that are consistent with equilibrium.

7 Ruling out more equilibria

7.1 No smooth equilibria

In this section we show that there cannot be any smooth equilibria in the case of imperfect altruism, i.e. $0 < \alpha + \alpha' < 2$.

First, note that there cannot be an equilibrium consisting of one WP-region by Proposition 4 in the paper. Second, there cannot be any equilibrium consisting of a single FT-region. To see why, suppose that she was the donor in such an equilibrium. Clearly, she would set $C = C_{WP}$ throughout. But he would not tolerate this when he has the power to do so. By Lemma 3 in the paper, he would set the transfer lower than her WP consumption at $P = 0$.

We are left with the possibility that there is an equilibrium consisting of a single NT-region. Such an equilibrium will generically not exist, as we show in the paper, since there are 4 boundary conditions (from the Party Theorem, Theorem 2 in the paper) for 2 first-order ODEs. Figure 6 shows numerical results: we start at $P = 0$ with the boundary conditions provided by the Party Theorem, then solve the two ODEs for consumption (coming from the EE) up to $P = 1$, and then compare to the boundary conditions $C(1) = C_{WP}$ and $C'(1) = C'_{lim}$ imposed by the Party Theorem. We find that throughout the parameter space, her consumption comes out too low and his too high.[12] As we move towards perfect altruism, the violations of the conditions become smaller but they are always there.

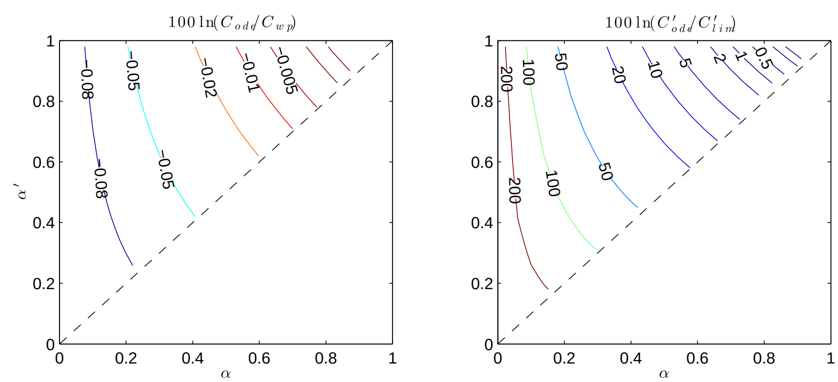

Figure 6: Boundary conditions for ODEs with one NT-region

We have now ruled out smooth equilibria that consist of *one* type of region. However, there might still be patched equilibria that are smooth on the boundaries in the sense that policies are continuously differentiable. Note that this cannot happen between FT and NT-regions since transfer functions are linear and positive on FT but zero on NT. It could however be that NT-regions smoothly turn into WP-regions (recall that transfers are indeterminate in those). Since there can only be one WP-region (Lemma 9 in the paper), it must be that this WP-region is then enclosed by two NT-regions which extend to the boundaries of the state space. As we will see in the following section, such an equilibrium does not exist.

7.2 No transfers-when-constrained equilibria with three regions

Although no transfers-when-constrained equilibria exist in which there is one MT-region $(0, 1)$, it is still conceivable that equilibria exist in which two NT-regions enclose a third region in the middle. Restricting our quest to symmetric equilibria in the case $\alpha = \alpha'$, we have found numerically that no such equilibria exist. By symmetry, the middle region can only be of NT, SS or WP type. We did not study cases with more than three regions.

[12] Note that by Proposition 5 in the paper we only need to consider α and α'. By symmetry, it is also sufficient to check only pairs $\{(\alpha, \alpha') : \alpha' \geq \alpha\}$

First consider the two cases in which the middle region is of SS- or WP-type. These two cases are special since the value functions are pinned down in these regions, see Lemmas 7 and 9 in the paper. We will try to construct an equilibrium as follows. The Party Theorem tells us the limiting consumption policies at $P = 0$. We can then solve for $\{C, C'\}$ from the EE, (29) in the paper, for NT-regions and check if one of the following two results occur: the value functions converge at some point to 1) the SS value functions or 2) the WP value functions. There is now one free boundary P_1, but two value-matching conditions. So we do not expect such an equilibrium to exist generically. Indeed, we have conducted computations for the special case of symmetric altruism ($\alpha = \alpha'$) on the entire range $\alpha \in (0, 1)$ and found that none of the two convergence results occurs for any α. The intuition for the non-existence results is as in the case of one NT-region. Since there are essentially two steady states bordering each NT-region (SS/WP and 0), the economy might converge to either one from each point in NT; it is extremely unlikely that the values of the two possibilities are the same for both players, which leads to conflicts that cannot be resolved in a deterministic setting.

Next, consider a potential NT-NT-NT equilibrium with two free boundaries $\{P_1, P_2\}$. Since value functions are not pinned down in NT-regions, it is more likely to find an equilibrium with this structure – we will now see why. As mentioned before, we will look for a symmetric equilibrium with $\alpha = \alpha'$ and boundaries $(P_1, 1 - P_1)$, so that $P_1 \in (0, \frac{1}{2})$ is the only free parameter.

Our equilibrium-construction strategy is as follows. Given the boundary conditions at $P = 0$ from the Party Theorem, we can solve the ODEs for consumption on NT up to the boundary P_1 for a given value of P_1. We then infer consumption policies on the other side of P_1 using Proposition 2. Using again the ODEs for NT-regions, we proceed to solve for consumption policies up to the point $P = \frac{1}{2}$. If his and her consumption policy are equal at this point, we have found an equilibrium. If not, we vary the free boundary P_1 until consumption policies are equal at $P = \frac{1}{2}$.[13] Note that this problem is exactly identified: we are varying one free parameter P_1 to meet one criterion at $P = \frac{1}{2}$.

Indeed, using this procedure we can find symmetric value functions and consumption policies for each $\alpha \in (0, 1)$ that satisfy the HJBs and Euler equations inside the regions and that are consistent with value-matching at P_1, see Figure 7 for an example.[14] The figure shows the consumption rates (C, C') in the upper-left and the value functions (V, V') in the upper-right panel. The vertical dashed line represents the boundary P_1. We see that value-matching is fulfilled at P_1. In the lower left panel, we have plotted players' consumption rates (c, c') out of their own assets (k, k'). As mentioned before, $\dot{P} > 0$ if and only if $c' > c$ in NT-regions. So the economy steers towards P_1 locally from both sides, so P_1 becomes an additional steady state. Again, we have the problem that there are "too many steady states" to which the economy can go. However, this time the

[13] Note that $C(\frac{1}{2}) = C'(\frac{1}{2})$ implies $V_P(\frac{1}{2}) = -V'_P(\frac{1}{2})$, and since then $\dot{P} = 0$ at $P = \frac{1}{2}$ it also implies $V(\frac{1}{2}) = V'(\frac{1}{2})$ by the HJB.

[14] We found, however, that for values of α above 0.4 the transfer motive becomes positive just left of P_1, which is already inconsistent with equilibrium.

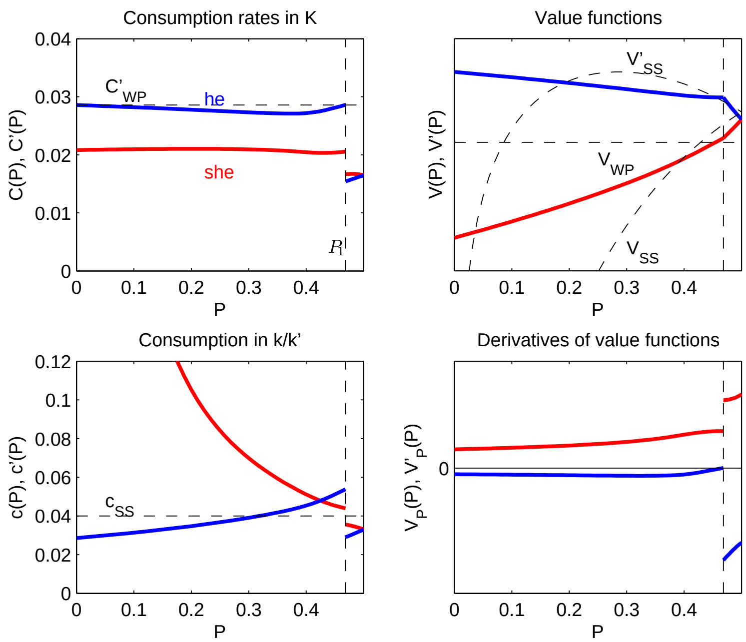

Figure 7: Symmetric equilibrium candidate: Transfers only in bankruptcy ($\alpha = \alpha' = 0.4, \rho = 0.04$).

technical reason for ruling out the equilibrium is different from the cases before: Corollary 1 tells us that there cannot be mutually best-responding policies at P_1 since it is an attracting boundary between two NT-regions.

The economic intuition for why the equilibrium breaks down at P_1 is the following. She (the player with the locally convex value function) wants to steer the economy away from P_1 once the boundary is left. She does not bear the downside consequences of over-consuming: if she is profligate and becomes poor, he will provide for her and give transfers eventually. If she is frugal at P_1, however, no transfers will flow and she will thus reap all the benefits from savings herself. For him, the situation is exactly the opposite. He is locally risk-averse (V' is locally concave) and tries to contain the economy at the boundary. If she over-consumes, he does not want to be the nice guy who is frugal, watches her party and then pays transfers in the end, so he prefers to also be profligate. This steers the economy back to P_1. If she is frugal, he also has incentives to be frugal since he will never have to give transfers.

If players had lotteries (or risky assets) at their disposition, then the locally risk-loving agent would make use of these at P_1.[15] This cannot be "counter-

[15] Laitner (1988) follows this route: he introduces a full set of lotteries into an altruistic OLG setting in which generations overlap for one period in order to remove non-concavities in the value functions.

acted" in any way by the risk-averse agent, unlike in the deterministic case, and enable the risk-loving agent to steer the economy away from P_1. We will see below how the introduction of a shock into the setting does indeed resolve the tensions: chance decides to which side the economy moves at the critical point.

7.3 No FT-to-SS equilibrium

The Prodigal-Son Dilemma tells us that there cannot be an equilibrium where a rich donor gives a mass transfer to a broke recipient and players are self-sufficient ever after. In this section, we show that there cannot exist an equilibrium either where the donor gives *flow* transfers to lift the poor player into self-sufficiency, i.e. an equilibrium where the region structure is FT'-SS(-FT).

For the special case where the SS-region covers the maximally-possible range (see Lemma 7 in the paper), we have the following theoretical result:

Proposition 3 *Suppose that $\alpha' > 0$. If there is a SS-region $\mathcal{P}_{SS} = (\frac{\alpha'}{1+\alpha'}, P_2)$, then there cannot exist a flow-transfer region $\mathcal{P}_{FT'} = [0, \frac{\alpha'}{1+\alpha'})$ in equilibrium. Analogously, supposing that $\alpha > 0$, if there is a SS-region $(P_{N-2}, \frac{1}{1+\alpha})$, then there cannot exist a flow-transfer region $\mathcal{P}_{FT} = (\frac{1}{1+\alpha}, 1]$.*

Proof: His value-matching condition at $\frac{\alpha'}{1+\alpha'}$ implies that her consumption in the FT-region is $C_{FT} = \frac{\alpha'\rho}{1+\alpha'}$. Then, her value-matching condition implies that $G'(\frac{\alpha'}{1+\alpha'}) = 0$. The ODE for transfers in a FT-region implies that $G'_P = 0$ throughout $\mathcal{P}_{FT'}$, which in turn implies $G'(0) = 0$. But this makes her consumption zero at $P = 0$, which means he is clearly not best-responding. ■

When we set the lower boundary of $\mathcal{P}_{SS}$ higher than $\frac{\alpha'}{1+\alpha'}$, we are not able to rule out this type of equilibrium using only paper and pencil. But when following the same route as in the lemma, this time solving the ODE for transfers numerically, we can find the implied $G'(0)$ for any parameter constellation and any location of the kink P_1 and then check if it is such that it can finance the recipient's consumption, i.e. $G'(0) \geq C(0)$. Proposition 1 in the paper tells us that we may fix $(\rho, r) = (1, 0)$ and vary only the tuple (α, α'). By Lemma 7 in the paper, we know that $P_1 \in [\underline{P}, \bar{P}] = \left[\frac{\alpha'}{1+\alpha'}, \frac{1}{1+\alpha}\right]$, since this is the maximally-possible range for a SS-region. Indeed, we find that the $G'(0)$ we calculate from the ODE is negative for all $P_1 \in [\underline{P}, \bar{P}]$ for all (α, α')-pairs, which is not even feasible. Figure 8 shows $\ln\left(\frac{-G'(0)}{C(0)}\right)$ from this numerical exercise, where we have parameterized the kink location by `KinkRatio` $= (P_1 - \underline{P})/(\bar{P} - \underline{P}) \in [0, 1]$. This shows that this type of equilibrium cannot exist either.

8 The Model with a shock

Let us consider the following variation to our model: instead of a safe return r, the agents now have access to a single asset with expected return r and variance σ^2 (over a unit of time). So agents face an idiosyncratic shock to assets,

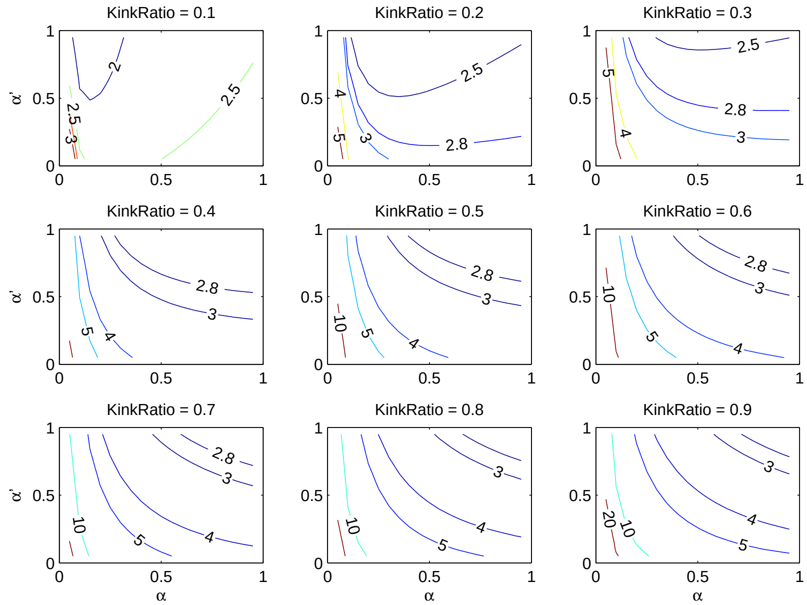

Figure 8: Implied $\ln\left(\frac{-G'(0)}{C(0)}\right)$ for FT-SS structure

which can be either interpreted as idiosyncratic savings risk or as shocks to expenditures (such as costs of house repair, medical treatment etc.). We first present the mathematical analysis of the model (which the reader may skip on the first reading) and then the economically relevant results.

8.1 Mathematical analysis

The laws of motion for assets turn into the following stochastic differential equations (SDEs):

$$dk_t = (rk_t - c_t)dt + \sigma k_t dB_t$$
$$dk'_t = (rk'_t - c'_t)dt + \sigma k'_t dB'_t,$$

where B_t and B'_t are uncorrelated Brownian motions. Applying the Ito rule to the functions $K(k,k') = k + k'$ and $P(k,k') = \frac{k}{k+k'}$, the laws of motion in P/K-space are

$$\begin{aligned} dK_t =& (r - C_t - C'_t)K_t dt + P_t K_t dB_t + (1-P_t)dB'_t, \\ dP_t =& \big[\underbrace{P_t C'_t - (1-P_t)C_t + P_t(1-P_t)(1-2P_t)\sigma^2}_{\equiv a_P}\big]dt + \\ &+ P_t(1-P_t)\sigma(dB_t - dB'_t), \end{aligned} \tag{20}$$

where we have introduced the notation a_P for the drift of P. We see that the second and third term on the right-hand side for the law of motion for P both vanish when $P \to 0$ or $P \to 1$, i.e. shocks have a negligible impact on the asset distribution when one player is very rich compared to the other.

Analogous to the deterministic case, we now integrate utility over time for arbitrary K-linear strategies in order to determine the functional form of the value function. Using the Ito rule, we find that

$$d\ln K_t = \Big(r - C(P_t) - C'(P_t) - \tfrac{\sigma^2}{2}\big[P_t^2 + (1-P_t)^2\big]\Big)dt + \sigma\big[P_t dB_t + (1-P_t)dB_t'\big].$$

Integrating and exponentiating we obtain

$$K_t = K_0 \exp\left(\int_0^t r - C(P_s) - C'(P_s) - \tfrac{\sigma^2}{2}[P_s^2 + (1-P_s)^2]ds + \right.$$
$$\left. + \sigma \int_0^t P_s dB_s + \sigma \int_0^t (1-P_s)dB_s'\right).$$

As in the deterministic model, we substitute this expression into the agent's criterion and obtain her value function:[16]

$$\begin{aligned}\tilde{V}^\sigma(P_0, K_0) =& E_0\left[\int_0^\infty e^{-\rho t}\Big[\ln\big(C(P_t)K_t\big) + \alpha \ln\big(C'(P_t)K_t\big)\Big]dt\right] = \\ =& \underbrace{\frac{1+\alpha}{\rho}\left(\ln K_0 + \frac{r}{\rho}\right)}_{=W(K_0)} - \\ & - E_0\left[\int_0^\infty e^{-\rho t}\Big(C(P_t) + C'(P_t) + \tfrac{\sigma^2}{2}[P_t^2 + (1-P_t)^2]\Big)dt\right] + \\ & + E_0\left[\int_0^\infty e^{-\rho t}\Big(\ln C(P_t) + \alpha \ln C'(P_t)\Big)dt\right].\end{aligned}$$

We add σ-superscripts to the value functions in order to distinguish them from the deterministic ones. Just as in the deterministic case, we see that the value function is additively separable in K and P. Indeed, the part in K takes the same functional form as in the deterministic case. We define $V^\sigma(P) \equiv \tilde{V}^\sigma(P, K) - W(K)$. Recall that the SDE (20) for P does not depend on K, so it valid to write $V^\sigma(\cdot)$ as a function of P only.

In order to build intuition, we will now heuristically derive the HJB for the stochastic case:

$$\tilde{V}^\sigma(P_t, K_t) = \alpha \ln\big(C'(P_t)K_t\big)\Delta t + \max_C \left\{ \ln\big(CK_t\big)\Delta t + e^{-\rho \Delta t} E_t \tilde{V}^\sigma(P_{t+\Delta t}, K_{t+\Delta t})\right\}.$$

[16] Here we use the fact that $E_0 \int_0^t f_s dB_s = 0$ for any B_t-measurable function f_t.

Using stochastic calculus, we expand the continuation value as follows:

$$\begin{aligned} e^{-\rho\Delta t}\tilde{V}^{\sigma}(P_{t+\Delta t}, K_{t+\Delta t}) \simeq& \tilde{V}^{\sigma}(P_t, K_t) - \rho\tilde{V}^{\sigma}(P_t, K_t) + \tilde{V}^{\sigma}_P \Delta P + \tilde{V}^{\sigma}_K \Delta K + \\ &+ \tilde{V}^{\sigma}_{PP}(\Delta P)^2 + \tilde{V}^{\sigma}_{KK}(\Delta K)^2, \end{aligned}$$

where we have used that $\tilde{V}^{\sigma}_{PK} = 0$. In order to use the Ito rule, observe that as Δt becomes small ΔK_t and ΔP_t are given by (20) and by stochastic calculus we have

$$\begin{aligned} (dK_t)^2 &= \sigma^2 K_t^2 [P_t^2 + (1-P_t)^2] dt, \\ (dP_t)^2 &= P_t^2 (1-P_t)^2 \tilde{\sigma}^2 dt, \\ (dK_t)(dP_t) &= 2K_t P_t (1-P_t)(P_t - \tfrac{1}{2})\sigma^2 dt. \end{aligned}$$

Using the closed forms for $\tilde{V}^{\sigma}_K = W_K$ and $\tilde{V}^{\sigma}_{KK} = W_{KK}$, we can then derive the following HJB:

$$\begin{aligned} \rho\tilde{V}^{\sigma}(P, K) =& (1+\alpha)\ln K + r\frac{1+\alpha}{\rho} + \alpha \ln C' - C'\left[\frac{1+\alpha}{\rho} - P\tilde{V}^{\sigma}_P\right] + \\ &+ \max_C \left\{ \ln C - C\left[\frac{1+\alpha}{\rho} + (1-P)\tilde{V}^{\sigma}_P\right]\right\} - \\ &- 2P(1-P)(P - \tfrac{1}{2})\sigma^2 \tilde{V}^{\sigma}_P + P^2(1-P)^2\sigma^2\tilde{V}^{\sigma}_{PP} - \\ &- \frac{1+\alpha}{\rho}\frac{\sigma^2}{2}[P^2 + (1-P)^2]. \end{aligned}$$

Notice that the last term (a mere function of P) stems from the Ito term in $\tilde{V}^{\sigma}_{KK}$ – this term captures risk aversion with respect to dynasty resources K. As noted before, diversification of dynasty assets minimizes this penalty when assets are equitably shared.

We now use the decomposition $\tilde{V}^{\sigma}(P, K) = W(K) + V^{\sigma}(P)$ and the fact that $\tilde{V}^{\sigma}_P = V^{\sigma}_P$, $V^{\sigma}_{PP} = \tilde{V}^{\sigma}_{PP}$ to obtain a second-order differential equation for $V^{\sigma}(P)$:

$$\begin{aligned} \rho V^{\sigma} =& \alpha \ln C' - C'\left[\frac{1+\alpha}{\rho} - PV^{\sigma}_P\right] - \frac{1+\alpha}{\rho}\frac{\sigma^2}{2}[P^2 + (1-P)^2] + \\ &+ \max_C \left\{ \ln C - C\left[\frac{1+\alpha}{\rho} + (1-P)V^{\sigma}_P\right]\right\} - \\ &- 2P(1-P)(P - \tfrac{1}{2})\sigma^2 V^{\sigma}_P + P^2(1-P)^2\sigma^2 V^{\sigma}_{PP}. \end{aligned} \tag{21}$$

We recognize the terms in C and C' from the deterministic HJB for V (equation 25 in the paper) and the constant in P as the penalty on uncertainty over K. The last two terms are somewhat harder to interpret. The term in V^{σ}_P arises since there is a trend in P unrelated to consumption decisions which stems from the fact that P is a concave function of k and k'. The term in V^{σ}_{PP} captures risk aversion with respect to the distribution of assets within the dynasty: if the value function is concave in P, then this term is negative, which means that she is averse to the risk that the asset distribution is reshuffled. This risk is independent of consumption decisions –since these cannot influence the agents' risk – and depends only on the exogenous variance σ^2.

We can proceed analogously for him to obtain his HJB for ${V'}^{\sigma}(P)$:

$$\begin{aligned}\rho {V'}^{\sigma} =& \alpha' \ln C - C\left[\frac{1+\alpha'}{\rho} + (1-P){V'}^{\sigma}_{P}\right] - \frac{1+\alpha'}{\rho}\frac{\sigma^2}{2}[P^2 + (1-P)^2] + \\ &+ \max_{C'}\left\{\ln C' - C'\left[\frac{1+\alpha'}{\rho} - P{V'}^{\sigma}_{P}\right]\right\} - \\ &- 2P(1-P)(P-\tfrac{1}{2})\sigma^2 {V'}^{\sigma}_{P} + P^2(1-P)^2\sigma^2 {V'}^{\sigma}_{PP}.\end{aligned}$$

His complete value function in two variables is $\tilde{V'}^{\sigma}(P,K) = W'(K) + {V'}^{\sigma}(P)$, where

$$W'(K) = \frac{1+\alpha'}{\rho}\left(\ln K + \frac{r}{\rho}\right).$$

Agents' optimal consumption rules are the same as in the deterministic case:

$$C^{\sigma *} = \left(\tfrac{1+\alpha}{\rho} + (1-P)V^{\sigma}_{P}\right)^{-1}, \qquad {C'}^{\sigma *} = \left(\tfrac{1+\alpha'}{\rho} - P{V'}^{\sigma}_{P}\right)^{-1}.$$

8.2 Results

We now calculate the solution of the game by backward-iterating on the HJB. We use a trinomial-grid method. The drift for P is a_P, where consumption is calculated from the FOC and transfers are set to zero in the interior of the state space (we check later if transfer motives are really negative). The variance of P is $P^2(1-P)^2 2\sigma^2$ (note that there are two Brownian motions). For the final guess on the value function, we adopt different specifications for what happens in the last interval Δt on the time grid (e.g. we assume that the static altruism game is played). The form of the final guess does not matter much, we always end up with the same solution to the game.

Figure 9 shows the resulting equilibrium.

The upper-left panel shows players' consumption C and C'. Around the middle of the state space they are very similar to SS consumption (the diagonal dashed lines). When the asset distribution is imbalanced, the donor's consumption is close to WP consumption (the horizontal dashed lines). The vertical dashed lines indicate the value of P where transfers would start to flow in a static altruism model. The discontinuity of the poor player's consumption at the constraint highlights point 3 of the Party Theorem (Theorem 2 in the paper).

As for the value functions in the lower-left panel, we see that risk-lovingness and risk-aversion follow the pattern already pointed out for the NT-NT-NT structure in Section 7.2. The lower-right panel show the dynamics of the state variable P. The drift of P is represented as a solid line, and 1-standard-deviation bands as dashed lines. In the neighborhood of $P = 0.5$, we see that the economy is basically stationary. However, the only absorbing states are $P \in \{0, 1\}$, which are the only points in the state space where the shocks do not influence the asset

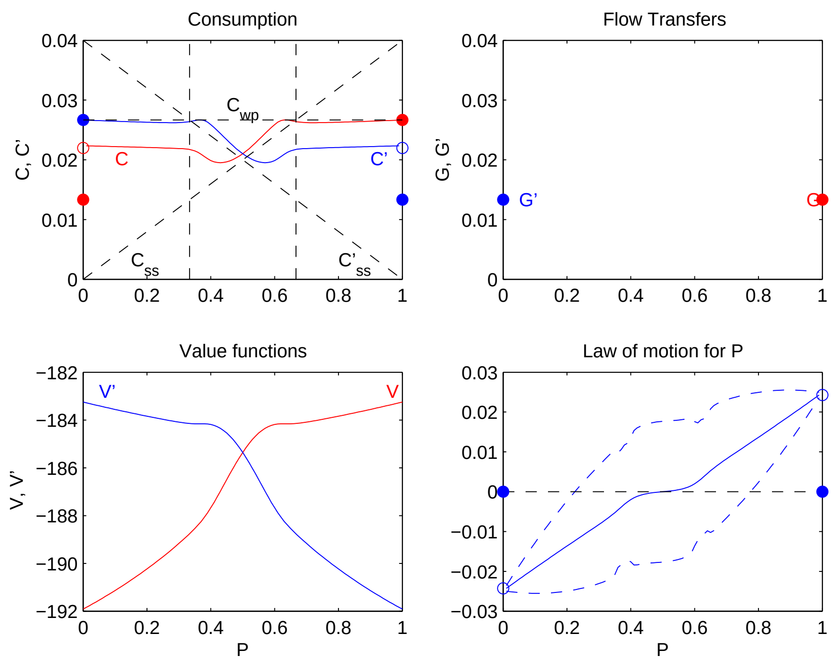

Figure 9: Transfer-when-constrained equilibrium with shock ($\alpha = \alpha' = 0.5$, $\rho = 0.04$, $r = 0$, $\sigma = 0.05$)

distribution. Once one player is broke, no shock can bring him away from there. We can see that when the initial asset distribution is imbalanced, immiseration of the poorer player is likely to occur.

The strength of this equilibrium, in addition to being empirically plausible and unique, is that it is stable with respect to the following objections that can be brought forward against the tragedy-of-the-commons-type equilibrium: it can be maintained under a finite horizon, and it survives the introduction of a shock as well as the introduction of in-kind transfers.

Technically, the problem of over-identification that we faced in the deterministic setting with a single NT-region disappears for the following reason. We still have four boundary conditions for consumption at $P \in \{0, 1\}$, but the ODEs for consumption on the NT-region are now of second instead of first order because we introduced Brownian motion (the shock). This makes the system exactly identified, so a unique equilibrium is the natural outcome. Economically speaking, the economy can now end up in either of the two steady states $\{0, 1\}$ from (almost) any starting point, so information from both sides enters the allocation for any $P \in (0, 1)$. As described above, randomness resolves the tensions that arose in the deterministic setting.

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