

# Inequality and Asset Prices\*

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## Abstract

This paper studies a class of incomplete-markets economies with shocks that are correlated in the sense that they are common to groups of individuals. We analyze three economies in a continuous-time setting. In each economy, a single security is traded by two agents: (i) an *asset* with an exogenous dividend, (ii) a short-term *bond*, and (iii) *capital* in a production economy. We show how to solve for equilibrium and find that in all three economies, when reasonably calibrated, the return of the security is lowest in times of high wealth concentration. We derive pricing equations in the asset and bond economy; we show that returns become infinitely elastic with respect to the wealth distribution when one agent is close to non-participation in the security market. In the capital economy, we characterize fluctuations in the aggregate capital stock and aggregate production that are solely due to re-distributive income shocks. These fluctuations are inefficient. Investment is highest in times of high wealth concentration, which helps wealth-poor households since it boosts their wages when most needed.

*Keywords:* Asset pricing; incomplete markets; inequality

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# 1 Introduction

Heterogeneous-agent economies with *idiosyncratic* shocks are a well-understood and widely-used class of models in macroeconomics. However, less is known about economies in which agents are subject to *correlated* shocks, i.e. shocks that hit entire groups of agents. Examples of such correlated shocks abound. Changes in technology or trade can make wages in entire industries, occupations or education groups rise or fall. Regions and even countries are subject to shocks from the realms of policy, nature and climate. Finally, policy changes such as re-distributive tax reforms make the disposable income of some population groups grow, while they reduce it for others.

This paper aims to further our understanding of incomplete-markets economies with correlated shocks, and to analyze how their workings differ from the idiosyncratic-shocks benchmark. We study a class of parsimonious models with two groups of agents, which allows us to derive interesting characterizations. We study three standard securities as the single asset in our model: (i) an *asset*, (ii) a *bond* and (iii) *capital*.<sup>1</sup> We show how correlated risk gives rise to an interesting interplay between the wealth distribution and asset returns in the three economies and compare the resulting equilibria.

This is clearly not the first paper on two-agent economies with incomplete markets. However, we can add to the existing literature because, unlike the bulk of existing papers, we work in a continuous-time setting with Poisson shocks. This has the following advantages: (i) Unlike in discrete-time settings, budget constraints only bind when an agent's asset holdings are exactly equal to the constraint, but never before. This allows sharper characterizations of pricing and consumption functions than in equivalent discrete-time settings. (ii) We can derive ordinary differential equations (ODEs), and partial differential equations (PDEs) for the non-stationary case, which yield pricing equations that are more transparent than in discrete time. (iii) Standard numerical solution methods apply for these ODEs and PDEs that are simpler and faster than the solution methods for discrete-time models.

Following the literature, we parameterize our three economies in a standard fashion.

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<sup>1</sup>More precisely, these securities are (i) a tree with constant dividend payments as in Lucas (1978), (ii) a short-term bond as in Huggett (1993), and (iii) capital in a neo-classical production economy as in Aiyagari (1994).

We find that a high concentration of wealth goes hand-in-hand with a low (expected) return to the security. The main force at work is that the asset's return has to fall in order to discourage the wealthy from building up excess demand for savings. The flip side is that supply of the security dries up once the wealth-poor group becomes constrained and ceases to participate in markets, which depresses the return. This mechanism has been mentioned in the literature, but we think that our paper provides a valuable and coherent overview on how it plays out in three different markets.<sup>2</sup>

In addition, we have the following new findings. In all economies, asset returns are most volatile and hardest to predict in times of extreme wealth concentration. This is due to the sudden drops and surges in asset demand that occur when one group exits or enters the market. Our calibrations indicate that the quantitative implications of our mechanism are of significant magnitude in all three economies; these results are robust to parameter changes.

The following findings are specific to subsets of the three economies. For the asset economy, we construct a pricing kernel that separates the effects of two forces: (a) consumption inequality, and (b) the likelihood of binding constraints. We find that the likelihood of binding constraints, and not consumption inequality, is responsible for the aforementioned relationship between asset prices and the wealth distribution.

We show that in the asset economy, the asset price becomes infinitely elastic with respect to a group's wealth share when approaching a binding constraint. Similarly, in the bond economy the bond yield drops down discontinuously at a binding constraint. We show how to deal with these peculiarities computationally. These results are of interest for applied researchers who use approximations to pricing functions to compute equilibria in incomplete-markets models, e.g. two-country trade models. Our findings indicate that it is important to allow for sufficient flexibility in the approximating functions in the neighborhood of binding constraints.

The capital economy has predictions that go beyond those of the asset and bond economies. As opposed to the other two economies, the wealth-rich group of agents may continue to accumulate capital even when the wealth-poor group is constrained since capital is not in

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<sup>2</sup>Den Haan (2001) describes this mechanism in a bond economy and Zhang (1997*b*) mentions it when studying a Lucas-tree economy, more on this in the literature review below. However, to the best of our knowledge, we are the first to study this mechanism in a capital economy.

fixed supply. Again, this makes returns lowest, and hardest to predict, in times of extreme wealth concentration. We find that aggregate investment exhibits two properties that are inefficient: not only is the capital stock always higher than in the efficient benchmark, but it also varies over time (whereas it is constant in the efficient steady state).<sup>3</sup> In other words, our economy displays fluctuations in aggregate economic activity that are caused solely by re-distributive shocks.<sup>4</sup> Over-accumulation of capital is especially severe in times of high wealth concentration. The wealth-rich drive up the capital stock by their savings, which depresses the interest rate. However, it also boosts wages. This benefits the wealth-poor since they derive the lion's share of their income from labor.

We now turn to a survey of the related literature. In the 1990s, a very lively literature asked if incomplete-markets economies could solve asset-pricing puzzles. Most of this literature used settings with a continuum of agents with idiosyncratic shocks (more on this below), while others approached it with a *finite* number of agents. Most closely related to our setting are Den Haan (2001) and Zhang (1997a,1997b). These papers study two-agent incomplete-markets economies with a single asset in discrete time, using computational methods. All of these models include aggregate shocks; we strip them out in order to focus more sharply on the wealth distribution. Den Haan (2001) studies a bond economy; his discussion of the relationship between the wealth distribution and bond yields (Section 3.1.1) mirrors ours, but we can add the following: We show that when trading is instantaneous, the bond yield displays a downward discontinuity when one agent becomes constrained. This is mirrored in a steep slope in Den Haan's numerical results (Figure 1). Zhang (1997a) also studies a bond economy; his focus is on an endogenously-determined borrowing constraint, which is chosen just tight enough so that the investor never prefers autarky to re-paying her debt. Zhang (1997b) repeats this exercise, but replacing the bond by a Lucas tree. His pricing functions are similar to ours, but we discuss in more depth the interplay of wealth distribution and asset returns (compare his Page 244 to our Section 2.7). Also, our pricing kernel and the analysis of prices at the constraint are new.<sup>5</sup>

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<sup>3</sup>The level of capital is too high in our economy due to the precautionary-savings motive; this finding is common to most incomplete-markets economies. However, in incomplete-markets with *idiosyncratic* shocks the capital stock is constant in steady state (and perfectly predictable off the steady state), whereas it varies in a stochastic fashion in our economy.

<sup>4</sup>This mirrors findings by Scheinkman & Weiss (1986), which we discuss below.

<sup>5</sup>Judd et al. (2000) study computational algorithms in a two-agent incomplete-markets economy with one

There are several other (computational) papers studying economies inhabited by a low *finite* number of agents, such as the two-agent economies of Telmer (1993), Lucas (1994) and Heaton & Lucas (1996). Den Haan (1996) studies settings with larger (finite) numbers of agents. Another area where incomplete-markets models with a small number of (usually two) agents are used is the international-finance and trade literature, see e.g. Stepanchuk & Tsyrennikov (2012). The aforementioned papers ask the question if incomplete markets can help us to understand the long-run properties of asset prices, such as the unconditional mean and variance of the equity premium and the risk-free rate. We, however, do not focus on the equity premium and other long-run properties, but study how (i.e. in which direction, how much and why) a single asset price interacts with the wealth distribution. Another difference is that those models have state spaces of high dimensionality, whereas our framework is extremely simple – it represents the wealth distribution by a single variable.

In terms of methodology, our paper is most closely related to Scheinkman & Weiss (1986). These authors study a two-agent continuous-time setting similar to ours in which one asset (a Lucas tree) is traded. The crucial difference to us is that in their framework the income-poor agent has no labor income at all. This income process together with an Inada condition on utility makes the household cling to the asset when income-poor; in equilibrium the borrowing constraint is never binding. Scheinkman & Weiss (1986) find that asset prices are increasing in the income-poor agent's wealth share. This is not the case in our asset economy, where they are U-shaped. This difference stems from another assumption they make: agents make a labor-leisure decision, disutility of labor being linear. This utility function fixes the productive (i.e. income-rich) agent's consumption at a constant level, which makes the model more tractable but leads to predictions that are different from the ones that we obtain in our more standard setting. As in our capital economy, re-distributive shocks cause aggregate economic activity to fluctuate in their setting. However, their mechanism is different from ours: in their setting variations in labor supply are responsible for aggregate fluctuations, whereas in ours it is the changing demand for precautionary savings.

Some papers have also looked at two-agent incomplete-markets economies with *two* assets, for example Marcet & Singleton (1999) and Judd et al. (2002). The portfolio-choice

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asset, but are not concerned with the economic mechanisms we are interested in.

problem makes both theory and computation a lot harder – even determining the economy’s state is not a simple task, as Marcet & Singleton (1999) remark. The peculiarities of the pricing functions that we identify in both the asset and the bond economy may contribute to the computational problems that Marcet & Singleton (1999) report for their algorithm. Our results are mirrored in some of the figures provided by Judd et al. (2002), who do not report problems with their algorithm. These authors do not discuss the economics relating to the wealth distribution since they are primarily interested in computational issues.

Finally, an entire industry has evolved that uses heterogeneous-agents economies with a *continuum* of agents subject to idiosyncratic shocks to explain asset prices. A non-exhaustive list of examples is Krusell & Smith (1997), Storesletten et al. (2007), Gomes & Michaelides (2008) and Guvenen (2009).<sup>6</sup> The key difference to our model is that shocks to individual agents are independent across agents in these models, whereas they are correlated in our setting. Independence of idiosyncratic shocks with a continuum of agents makes them wash out at the aggregate level, and asset prices are time-invariant in such economies unless there are aggregate shocks. When adding aggregate shocks, idiosyncratic shocks still wash out in a reasonably calibrated economy, as Krusell & Smith’s (1998) much-used approximate-aggregation result asserts: prices can be forecast very accurately using only the mean asset holdings of all agents. Our economy is a stark counterexample to approximate aggregation: the wealth share held by the rich group of agents (a measure of inequality) is key for predicting asset returns.<sup>7</sup>

The remaining paper is organized as follows. Section 2 studies the asset economy, Section 3 the bond economy, and Section 4 the capital economy. Section 5 discusses the numerical results of a standard parameterization from the three economies. Section 6 concludes with a discussion on how idiosyncratic and correlated shocks could be combined in future research.

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<sup>6</sup>Hommes (2006) provides a survey of heterogeneous-agents models in finance. Krueger & Lustig (2010) give conditions under which market incompleteness is irrelevant for the price of aggregate risk in heterogeneous-agents models with idiosyncratic shocks (which is key for the size of the equity premium). The difference of their question to ours is that we study the *level* of asset prices, that shocks to individuals are correlated and not idiosyncratic in our model, and that our setting has no aggregate risk.

<sup>7</sup>Yet another approach to heterogeneous-agents asset pricing, taken by Alvarez & Jermann (2001), are complete markets with endogenous solvency constraints.

## 2 Asset economy

Following Scheinkman & Weiss (1986) and Kehoe & Levine (2001), we write down the simplest-possible economy with two infinitely-lived agents facing income risk. There is no aggregate risk. We will discuss these simplifying assumptions below.

Time is continuous:  $t \in [0, \infty)$ . Two classes of agents of the same measure, indexed by  $i \in \{1, 2\}$ , inhabit the economy. There is a single consumption good and one asset – a Lucas tree – in the economy, which is in unit fixed supply. The asset yields a constant dividend stream  $q \in (0, 1)$ . Agents' asset position  $a_t$  must satisfy  $a_t \geq -\bar{A}$ , where  $\bar{A} \geq 0$  is an exogenous borrowing limit.<sup>8</sup> We will often set  $\bar{A} = 0$ , which is equivalent to a no-short-selling constraint. The asset's price in terms of the consumption good is denoted by  $P_t$ . Agent 1's labor income  $y_t$  follows a two-state Poisson process, which switches between the states  $y_l \leq y_h$  at hazard rate  $\eta > 0$ .<sup>9</sup> Agent 2 faces a labor-income process with the same realizations  $\{y_l, y_h\}$  and the same switching hazard  $\eta$  as agent 1, but the realizations have perfect negative correlation with those of agent 1: when agent 1 receives  $y_h$ , then agent 2 receives  $y_l$ , and vice versa. The economy's aggregate endowment is constant and normalized to 1:  $q + y_h + y_l = 1$ . Both agents have standard preferences over consumption  $c_t$ :

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt,$$

where  $\rho > 0$  and where we assume  $u' > 0$ ,  $u'' < 0$  and  $\lim_{c \rightarrow 0} u'(c) = \infty$ .

### 2.1 Discussion of our modeling assumptions

The assumption that income shocks are perfectly negatively correlated across the two groups may seem restrictive at first. However, this is what arises naturally if one abstracts away from aggregate shocks. To see this, consider an arbitrary stochastic process for earnings of the two groups. If we divide each group's earnings by total (labor) earnings in the economy in order to strip out aggregate effects, we end up with the labor-income share

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<sup>8</sup>Since agents are symmetric, we will denote by  $a_t$  the asset position of agent 1, omitting the index  $i = 1$ . The analysis for agent 2 is always analogous.

<sup>9</sup>We focus on a symmetric transition probability to simplify the exposition. It is straightforward to extend the analysis to general Poisson processes, also with a larger number of states.

of each group. Since they are shares, the two shares add up to one each period and the gain of one group must be the loss of the other. The two shares thus have perfect negative correlation. Our approach can be easily extended to income processes that have a different correlation structure, for example positive correlation. We opt for this simpler specification because we want to keep the number of states low and want to focus the analysis on the effects of re-distributive shocks without being confounded by aggregate shocks.

Similarly, the reader may wonder if our results extend to a setting with more than two agents. We choose a two-agent model because it has the advantage that we can summarize the wealth distribution in a single variable. This also enables us to characterize equilibrium by ordinary differential equations in this variable. If we extend the setting to three agents, our tools can still be applied, but the wealth distribution would now be captured by two variables from the two-dimensional simplex. Equilibrium would be characterized by partial differential equations on the simplex, making the analysis more cumbersome and less transparent. Also, the essence of our arguments will not rely on the two-agent structure, so we expect the same results to go through in settings with more than two agents.

## 2.2 The agent's problem

The budget constraint at  $t$ , given the inherited asset position from  $t - \Delta t$ , is

$$c_t \Delta t + P_t a_t = y_t \Delta t + q a_{t-\Delta t} + P_t a_{t-\Delta t}.$$

Dividing by  $\Delta t$  and taking limits as  $\Delta t \rightarrow 0$ , we obtain the flow budget constraint

$$c_t + P_t \dot{a}_t = y_t + q a_t, \tag{1}$$

where we have denoted the drift in the asset position by  $\dot{a}_t = \lim_{\Delta t \rightarrow 0} (a_t - a_{t-\Delta t}) / \Delta t$ .

We look for an equilibrium in which all agents of the same type choose the same policy. The aggregate state of the economy is  $(y_t, A_t)$ , where  $A_t \in [-\bar{A}, 1 + \bar{A}]$  denotes the asset position of a typical type-1 agent. Note that we have to give the agent the possibility to deviate in his asset choice  $a_t$  from the aggregate position of his group  $A_t$  when writing

down the agent's problem.<sup>10</sup> The problem for a group-1 agent is to choose a contingent consumption process  $\{c_t\}_{t=0}^{\infty}$  in order to solve

$$\begin{aligned}
& \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt & (2) \\
& \text{s.t.} \quad \dot{a}_t = (y_t + qa_t - c_t)/P_t, \\
& \quad \quad a_t \geq -\bar{A}, \\
& \text{given} \quad \dot{A}_t = \underbrace{(y_t + qA_t - C_t)/P_t}_{\equiv d^A(y_t, A_t)}, \\
& \quad \quad C_t = C(y_t, A_t), \\
& \quad \quad P_t = P(y_t, A_t), \\
& \quad \quad a_0 = A_0.
\end{aligned}$$

The first constraint is the law of motion for the individual agent's asset, the second constraint limits borrowing. The agent takes as given a *perceived law of motion*  $d^A(y_t, A_t)$  for the aggregate variable  $A_t$ , which is implied by a pricing function  $P(y_t, A_t)$  and a consumption rule  $C(y_t, A_t)$  for the typical agent 1. The individual takes the functions  $P$  and  $C$  (and thus  $d^A$ ) as exogenously given. The problem for agent 2 is the same as the one given in (2) except that income  $y_t$  in the law of motion is replaced by  $1 - q - y_t$  and initial assets by  $a_0 = 1 - A_0$ .

We will now cast the agent's problem in (2) as a dynamic-programming problem with state  $(y, A; a)$ . The Hamilton-Jacobi-Bellman equation (HJB) for agent 1 is

$$\rho V = \max_c \left\{ u(c) + \frac{y + qa - c}{P} V_a \right\} + \eta V_y + d^A V_A, \quad (3)$$

where we denote partial derivatives by subscripts. For notational convenience, we also introduce the "discrete derivative"  $V_y \equiv V(y', \cdot) - V(y, \cdot)$ , in which  $y$  denotes the current state of income and  $y'$  is the alternative income state. The arguments of the value function

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<sup>10</sup>This is as in the big- $K$ -little- $k$  trick in the text-book case of the neo-classical growth model. If we wrote a problem where the individual controls the aggregate state  $A_t$ , then this would give the individual influence over prices. This may be interesting for applications where agents possess market power, but is not of interest in our context.

are suppressed for better readability.

The first-order condition (FOC) for the consumption-savings choice is

$$P u_c(c^*) = V_a \quad (4)$$

and asserts that the marginal value of owning the asset  $V_a$  has to be equal to the value of selling a marginal unit at price  $P_t$  and consuming it. When  $a_t = -\bar{A}$ , then the agent may be constrained and (4) becomes an inequality. If the agent's marginal utility of consuming his flow income is larger than the marginal value from buying the asset, i.e.  $P_t u_c(y_t - q\bar{A}) > V_a$ , then the agent is constrained and optimal consumption is  $c^* = y - q\bar{A}$ .

At this point it is convenient to introduce the *infinitesimal generator*, which is a partial differential operator that tells us about the expected growth of any smooth function  $f(\cdot)$  defined on the agent's state  $(y, A; a)$ :

$$\begin{aligned} \mathcal{A}f &= \lim_{\Delta t \rightarrow 0} \mathbb{E}_t \left[ \frac{f(y_{t+\Delta t}, A_{t+\Delta t}; a_{t+\Delta t}) - f(y_t, A_t; a_t)}{\Delta t} \right] = \\ &= \eta f_y + d^A f_A + \frac{y + qa - c^*}{P} f_a, \end{aligned} \quad (5)$$

where  $c^*$  is the optimal consumption rule for agent 1.

Using the infinitesimal generator, we can re-state the HJB (3) as follows:

$$\mathcal{A}V = \rho V - u(c^*). \quad (6)$$

Now, taking derivatives of the HJB (3) with respect to  $a$  and using the FOC (4), we find the Euler equation

$$\rho V_a = \frac{q}{P} V_a + \frac{y + qa - c}{P} V_{aa} + \eta V_{ya} + d^A V_{aA}.$$

At this step it is important to note that  $P_a = 0$  and  $d_a^A = 0$ , i.e. the individual agent can neither influence prices nor the aggregate law of motion.<sup>11</sup> Using the infinitesimal generator, we can re-write the Euler equation as

$$\mathcal{A}V_a = \eta (V_a)_y + d^A (V_a)_A + \frac{y + qa - c}{P} (V_a)_a = \left( \rho - \frac{q}{P} \right) V_a.$$

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<sup>11</sup>... this is the important part of the big- $K$ -little- $k$  trick!

Finally, using the FOC (4), we obtain the Euler equation in its standard form:

$$\frac{\mathcal{A}[Pu_c(c)]}{Pu_c(c)} = \rho - \frac{q}{P}. \quad (7)$$

It says that the agent's marginal valuation of the asset  $Pu_c(c)$  follows a martingale when we adjust for discounting  $\rho$  and the dividend stream  $q$ . More precisely, the percentage growth rate of the marginal valuation of the asset,  $Pu_c(c)$ , grows at rate  $\rho$  minus the asset's dividend-price ratio. By symmetry, this Euler equation must also hold for the consumption plan of type-2 agents.

Note that the Euler equation (7) is also valid for the case of complete markets, which is an important benchmark for our analysis. With complete markets, agents enjoy perfect consumption insurance since there is no aggregate income risk. Thus  $c$  is constant, and the left-hand side of the Euler equation is zero. This implies that the complete-markets price is constant and equal to the risk-neutral valuation of the asset:  $P_{cm} = d/\rho$ .

### 2.3 Equilibrium definition

In equilibrium, privately-chosen consumption must equal aggregate consumption of the respective group and the asset market must clear. We define:

**Definition 1 (Equilibrium in asset economy)** *A competitive equilibrium for the asset economy consists of functions  $c^1(y, A)$ ,  $c^2(y, A)$ ,  $C(y, A)$  and  $P(y, A)$  such that*

1. *the stochastic process  $c^i(y_t, A_t)$  solves agent  $i$ 's problem (2) given the perceived law of motion  $C(\cdot)$  and  $P(\cdot)$ , for  $i \in \{1, 2\}$ ;*
2. *agents' expectations are consistent and markets clear:*

$$\begin{aligned} c^1(y, A) &= C(y, A), \\ c^2(y, A) &= 1 - C(y, A). \end{aligned}$$

The consistency conditions directly imply market clearing for the consumption good and, by Walras' Law, also market clearing in the asset market.

## 2.4 Euler equations

Our strategy to find an equilibrium is built on the insight that aggregate consumption  $C$  (for group 1) and  $1 - C$  (for group 2) must fulfill Euler equations. Formally, we use  $c^1 = C$  and  $c^2 = 1 - C$  in the Euler equation (7) to write

$$\frac{\mathcal{A}[Pu_c(C)]}{Pu_c(C)} = \underbrace{\eta \left[ \frac{P'u_c(C')}{Pu_c(C)} - 1 \right]}_{\text{jump in } y} + \underbrace{\frac{P_A \dot{A}}{P} + \frac{u_{cc}(C)}{u_c(C)} C_A \dot{A}}_{\text{no jump}} = \rho - \frac{q}{P}, \quad (8)$$

$$\frac{\mathcal{A}[Pu_c(1 - C)]}{Pu_c(1 - C)} = \eta \left[ \frac{P'u_c(1 - C')}{Pu_c(1 - C)} - 1 \right] + \frac{P_A \dot{A}}{P} - \frac{u_{cc}(1 - C)}{u_c(1 - C)} C_A \dot{A} = \rho - \frac{q}{P},$$

where the prime ( $'$ ) indicates values a variable takes under the alternative income state.

The first equation states that the percentage change in the valuation of the asset by a typical agent 1 is composed of two terms. The first term captures reversals in income  $y$ , in which case both  $P$  and  $C$  change discretely. The second group of terms captures “normal times” when there is no reversal in  $y$ . The change in valuation in normal times is just the sum of the percentage change in the asset price and the percentage change in marginal utility, which in turn is given by the coefficient of absolute risk aversion,  $u_{cc}/u_c$ , times the time change in consumption,  $\dot{C} = C_A \dot{A}$ .

The two Euler equations given in (8) give us four first-order ordinary differential equations (ODEs) for the functions  $C(y_l, \cdot)$ ,  $C(y_h, \cdot)$ ,  $P(y_1, \cdot)$ ,  $P(y_2, \cdot)$  defined on the domain  $(-\bar{A}, 1 + \bar{A})$ . In order to solve this system of equations we need four boundary conditions, which we will now obtain from the optimality conditions at the constraint. We guess an equilibrium where agent 1 is constrained at  $A = -\bar{A}$  only when receiving the low labor income. In this case, only agent 2 holds the asset and thus only agent 2’s Euler equation holds. Similarly, we guess that agent 2 is constrained at  $A = 1 + \bar{A}$  only when  $y = y_h$ , with only agent 1’s Euler equation holding. This gives us the four boundary conditions for the

four ODEs contained in (8):

$$\begin{aligned} C &= y_l + q(-\bar{A}) && \text{for } (y, A) = (y_l, -\bar{A}), \\ \rho - \frac{q}{P} &= \eta \left[ \frac{P' u_c(1 - C')}{P u_c(1 - C)} - 1 \right] && \text{for } (y, A) = (y_l, -\bar{A}), \end{aligned} \quad (9)$$

$$\begin{aligned} C &= y_h + (1 + \bar{A})q && \text{for } (y, A) = (y_h, 1 + \bar{A}), \\ \rho - \frac{q}{P} &= \eta \left[ \frac{P' u_c(C')}{P u_c(C)} - 1 \right] && \text{for } (y, A) = (y_h, 1 + \bar{A}). \end{aligned} \quad (10)$$

It turns out that finding a solution to this system of ODEs is not straightforward. We were not able to find a closed-form solution, and even with numerical methods we encountered problems. Standard methods run into trouble because the functions' slope approaches infinity at the boundaries of the state space. We will later analyze this phenomenon in more detail; we first discuss the other properties of equilibrium in order to facilitate the exposition.

## 2.5 Computing equilibria

Our computational approach is to re-write the problem as a finite-horizon economy. We then backward-iterate on consumption and pricing functions until they become time-invariant.<sup>12</sup> To do this, we need to introduce time as an additional state variable. This turns the Euler equations from ODEs into partial differential equations (PDEs). The advantage of this method is that the derivatives in the time dimension are always finite, unlike the derivatives in the  $A$ -dimension in the stationary case.

Let  $T > 0$  be the end date for the economy, such that  $t \in [0, T]$ . The value functions, consumption and pricing rules now have  $t$  as an additional argument. It turns out, however, that we only have to adapt the infinitesimal generator given by Equation (5) to this new

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<sup>12</sup>Note that this approach automatically excludes potential bubble equilibria. We also looked for bubble solutions working with the stationary ODEs but could not find any such equilibria. Specifically, we couldn't find a stationary equilibrium in the case of fiat money ( $q = 0$ ) as it exists in Bewley models. This could either be because such equilibria do not exist or may be due to the aforementioned computational difficulties.

setting. Re-define, for any smooth function  $f(t, y, A; a)$ :

$$\mathcal{A}f = f_t + \eta f_y + d^A f_A + \frac{y + qa - c^*}{P} f_a,$$

where  $f_t$  is the partial derivative of  $f$  in  $t$ . Using this re-defined operator, the HJB for the individual agent is still as in (6) and the Euler equation is still as in (7).

In order to solve for the consumption dynamics, we multiply both Euler equations by  $\frac{1}{2}$  and subtract them from each other in order to eliminate terms in  $P_A$ , which yields

$$\underbrace{C_t + C_A \dot{A}}_{\equiv \dot{C}} = \frac{\frac{\eta P'}{2P} \left( \frac{u_c(C')}{u_c(C)} - \frac{u_c(1-C')}{u_c(1-C)} \right)}{\frac{1}{2}\alpha(C) + \frac{1}{2}\alpha(1-C)}, \quad (11)$$

where  $\alpha(c) = -u_{cc}/u_c(c)$  is the coefficient of absolute risk aversion and  $\dot{C} \equiv \frac{d}{dt}C$  denotes the infinitesimal time change in consumption in normal times. Equation (11) thus allows us to calculate  $C_t$  given knowledge of the function values  $(C, C'; P, P')$  and the derivative  $C_A$  at any point of the state space.<sup>13</sup> From (11), we see that the sign of  $\dot{C}$  is determined by the parenthesis in the numerator. Whenever  $C' < C$ , i.e. whenever agent 1 faces downward consumption risk upon a reversal, then  $\dot{C} > 0$ , i.e. agent 1's consumption rises in normal times. Whenever agent 1 would increase his consumption upon a reversal ( $C' > C$ ), however, the opposite is true and his consumption falls in normal times ( $\dot{C} < 0$ ). We recognize the familiar martingale property of the optimal consumption process. Furthermore, the more likely a reversal (i.e. the higher  $\eta$ ), the more consumption has to rise in normal times for a given consumption risk  $|C - C'|$ .

We still have to find the updating rule for the pricing function. To do this, we add the two Euler equations (again multiplying each by  $\frac{1}{2}$ ) to obtain

$$\frac{\dot{P}}{P} = \rho - \frac{q}{P} + \eta \left[ 1 - \frac{P'}{P} \left( \frac{1}{2} \frac{u_c(C')}{u_c(C)} + \frac{1}{2} \frac{u_c(1-C')}{u_c(1-C)} \right) \right] + \left[ \frac{1}{2}\alpha(C) - \frac{1}{2}\alpha(1-C) \right] \dot{C}, \quad (12)$$

where  $\dot{P} \equiv \frac{d}{dt}P = P_t + P_A \dot{A}$ . Given the function values  $(C, C'; P, P')$ ,  $P_A$  and our solution for  $\dot{C}$  from (11), this equation allows us to calculate  $P_t$  and to update the pricing function.

<sup>13</sup>Setting  $C_t = 0$  gives us two ODEs for  $\{C, C'\}$  in the time-invariant case.

In (12), we see that the coefficients of risk aversion and the marginal-utility ratios play key roles for the determination of asset prices. The term  $\rho - q/P$  on the right-hand side is entirely standard – indeed, we recognize the only term that survives in the complete-markets case when we switch off consumption risk. The second group of terms shows that  $\dot{P}$  is decreasing in the marginal-utility ratios  $u_c(C')/u_c(C)$  and  $u_c(1 - C')/u_c(1 - C)$ . The more agents fear a reversal (i.e. the more marginal utility rises if a jump occurs), the more they appreciate the insurance the asset provides. Agents thus require lower returns to the asset in normal times, so  $\dot{P}$  is low. Similarly, an appreciation of the asset upon a reversal ( $P'/P$ ) makes the asset attractive and thus lowers  $\dot{P}$ . Finally, the last term on the right-hand side captures the evolution of marginal utility in normal times. If  $\dot{C}$  is positive and  $\alpha(C)$  is large, then marginal utility falls by a lot in normal times. The asset is not valuable as insurance in this scenario, and  $\dot{P}$  has to be high for agents to demand the asset.

In order to solve the model numerically we discretize time into small intervals  $\Delta t$ . At the end of the horizon  $T$  the asset is worthless. Thus, agents' consumption is given by their endowment plus dividends from the share of the tree they own, i.e.  $C(T - \Delta t, A, y) = qA + y$ . This means that the tree must be worth exactly the number of fruits that fall from it in the time interval  $[T - \Delta t, T]$  and we approximate  $P(T - \Delta t, A, y) = q\Delta t$ . Now we can use standard PDE solution techniques on a grid to solve from  $T - \Delta t$  backwards. The consumption and pricing functions are updated backward in time using (11) and (12). At the boundaries of the state space  $A \in \{-\bar{A}, 1 + \bar{A}\}$  we enforce the no-short-selling limits and update prices using Equations (9) and (10) when the constraints bind. Confirming our guess, we find that the constraint only binds if an asset-poor agent is also income-poor. The algorithm is very fast and reliable and converged for wide range of parameter configurations.

## 2.6 Parameterization

We parameterize the economy in a standard fashion. We choose a CRRA utility function  $u(c) = c^{1-\gamma}/(1-\gamma)$  with coefficient of relative risk aversion  $\gamma = 2$  and rate of time preference  $\rho = 0.035$ . We interpret the tree as the capital stock of the economy and set  $q = 0.3$  in order to have a realistic capital-income share. To make our results comparable to the literature, we parameterize the income process following the two-state calibrations

of Den Haan (2001) and Zhang (1997a), who both build on the estimates of Heaton & Lucas (1996) from the Panel Study of Income Dynamics (PSID). To obtain a ratio  $y_h/y_l = 1.2456/0.7544$  as used in the aforementioned papers, we set  $y_l = 0.264$  and  $y_h = 0.436$ . The income-change hazard is  $\eta = 0.25$ .

## 2.7 Equilibrium analysis

Figure 1 shows the equilibrium price and consumption functions, which are reminiscent of Zhang (1997b).

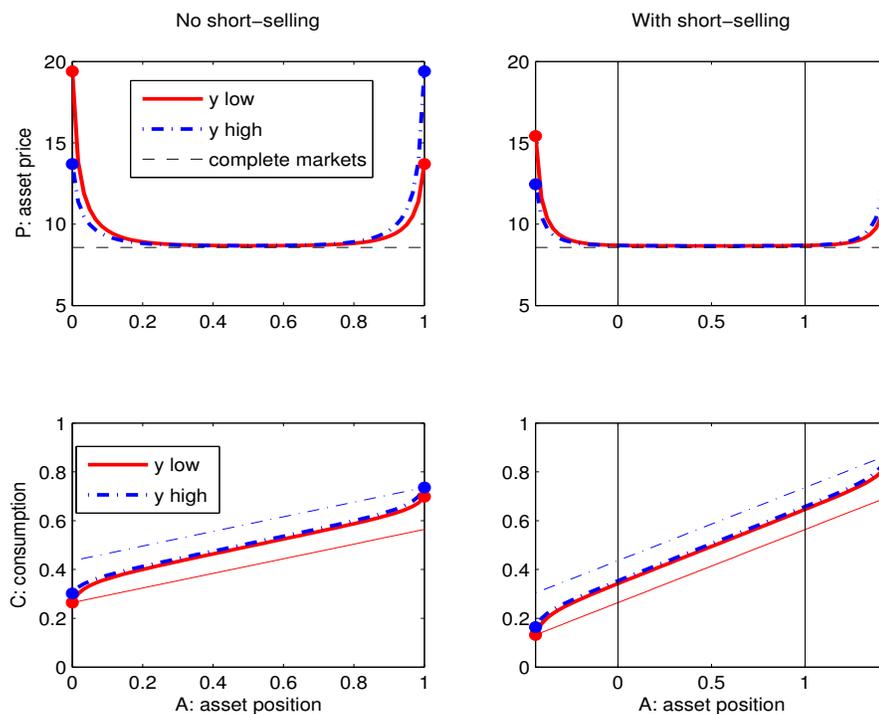


Figure 1: Asset economy

The thick lines correspond to equilibrium asset prices (top graphs) and consumption functions (bottom graphs) with  $\bar{A} = 0$  (left-hand side) and with  $\bar{A} = 0.44$  (right-hand side). The horizontal dashed line in the top two figures is the complete-markets price,  $P_{cm}$ . The thin lines in the bottom two figures depict agents' total flow income (the sum of labor and asset income) in each state.

The left panel depicts the case in which there is a no-short selling constraint, i.e.  $\bar{A} = 0$ ,

whereas in the right panel short-selling is allowed up to  $\bar{A} = 0.44$ .<sup>14</sup> Along the horizontal axis is the fraction of the tree owned by agent 1,  $A$ . The red solid lines indicate the state where agent 1 has the low income realization  $y_l$ , and the blue dash-dotted lines the state where he has the high income realization  $y_h$ .

Consider first the consumption functions depicted in the two bottom figures. Their features are standard for incomplete-markets models. Consumption is increasing in both the agent's wealth and income. When income-rich, agent 1 consumes less than his flow income  $y_h + qA$ , the blue dashed line, in order to accumulate precautionary savings. When income-poor, however, he consumes more than his flow income,  $y_l + qA$  (the thin solid red line), running down the buffer stock of wealth in anticipation of switching back to the high-income realization.

After a long spell of low income realizations, agent 1 reaches the lower bound of asset holdings,  $-\bar{A}$ . The economy then stays at this point until an income reversal occurs. In this situation, agent 1 is constrained and consumes his entire flow income,  $y_l - q\bar{A}$ . Agent 2 is in the best possible state – she has the high income and owns the entire tree – and consumes her total flow income.

Once an income reversal occurs in this extreme situation, agent 1's consumption jumps up discretely and he starts saving. On the other hand, agent 2's income level is now lower and she is dissaving. These new dynamics drive the wealth share gradually away from  $-\bar{A}$ . After a long spell of agent 1 receiving the high income, the economy ends up at  $A = 1 + \bar{A}$ , agent 2 being constrained.

The two top figures show the equilibrium pricing functions. Because markets are incomplete, agents use the asset to self-insure against consumption risk, driving up the asset's value above the complete-markets price  $P_{cm} = q/\rho$  (the dashed line). This need for self-insurance is mitigated when we allow for short-selling, and the asset price moves closer to the complete-markets price. Asset prices are highest when one agent is constrained (having low income at the lower bound for assets), coinciding with maximal consumption inequality.

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<sup>14</sup>We chose the constraint  $\bar{A}$  to be one half of the *maximal borrowing limit* in the second scenario. We define this maximal limit to be the short position  $\bar{A}_{max}$  that still allows the agent to honor the dividend payments arising from his short position when having the low income. Agent 1 can afford the dividend payment when constrained if and only if  $\bar{A}q \leq y_l$ , from which we find  $\bar{A}_{max} = y_l/q$ .

Why are prices highest when inequality is maximal? The reason is that the constrained agent ceases to participate in the asset market, and thus asset supply drops down once the poor agent sells off the last unit of his assets. The price of the asset then has to adjust such that the unconstrained agent is willing to maintain his current asset position and to consume exactly his flow income,  $y_h + q\bar{A}$ . The price rises, lowering the asset's expected return and thus the attractiveness of further savings for the rich agent. There is no trade at this point. Furthermore, once an income reversal occurs, the wealthy agent loses wealth since the asset price drops precipitously; this makes the wealthy desire even more insurance against a reversal, which pushes up the asset price further.

The price at this extreme state feeds back into the state space, which explains the U-shaped pricing function – we will discuss the intuition behind this shape in more detail when presenting our pricing kernel. Fixing  $A$ , we see that asset prices are higher when the wealth-rich agent is also income-rich. In this situation, wealth inequality is on the rise, and the economy moves closer to the constraint. Income inequality is also high: the agent with the high labor income also has more capital income. We conclude that asset prices are increasing in both wealth and income inequality, and that prices are maximal when wealth inequality, income inequality, and consumption inequality reach their maximal level.

We now turn to the volatility of prices, which we see to be a monotone function of the vertical distance between the two pricing functions:

$$\begin{aligned} \text{var}(P) &= \lim_{\Delta t \rightarrow 0} \mathbb{E}_t \frac{(P_{t+\Delta t} - P_t)^2}{\Delta t} = \\ &= \lim_{\Delta t \rightarrow 0} \mathbb{E}_t \frac{(1 - \eta\Delta t)(\dot{A}P_A\Delta t)^2 + \eta\Delta t(P' - P)^2 + o(\Delta t)}{\Delta t} = \eta(P' - P)^2. \end{aligned}$$

Volatility is also increasing in inequality and becomes maximal when the wealth distribution is entirely concentrated. The reason for this are the large swings in asset supply that occur when one agent leaves the market.

Finally, we can study the effect of a loosening in short-selling constraints comparing the left- and right-hand side of Figure 1. Such a loosening is commonly interpreted as financial-markets development. We find that the (time-series) consumption risk, as measured by the vertical distance between the consumption functions, is lower in the economy with looser constraints. This lower consumption risk not only exerts downward pressure on the level

of asset prices, it also mutes the sensitivity of asset prices to inequality and volatility at the constraints. Because insurance is better, constraints bind less frequently, thus the scenario of binding constraints affects prices less. This mechanism will become even clearer when considering the pricing kernel that we now introduce.

## 2.8 Pricing kernel

Since markets are incomplete, a unique stochastic discount factor (SDF) does not exist. But it turns out that there is a particularly useful SDF, which sheds light on the connection between asset prices, consumption inequality, and binding constraints.

For any interior point  $A \in (-\bar{A}, 1 + \bar{A})$  and  $\Delta t$  small enough, we can take the average over both agents' Euler equations (8) and integrate stochastically over time to obtain

$$\Gamma_t P_t = \mathbb{E}_t \int_t^{t+\Delta t} e^{-\rho(s-t)} q \Gamma_s ds + e^{-\rho\Delta t} \mathbb{E}_t [\Gamma_{t+\Delta t} P_{t+\Delta t}], \quad (13)$$

where we define  $\Gamma$  to be average marginal utility:

$$\Gamma_t \equiv \frac{1}{2} u_c(C_t) + \frac{1}{2} u_c(1 - C_t).$$

$\Gamma_t$  will form part of our SDF. However, once we include pricing at the constraints – Euler equations (9) and (10) –, we will see that we need an additional element. For now, however, it is useful to focus on  $\Gamma_t$  because it highlights the role of consumption inequality.

As long as marginal utility is convex (i.e. if preferences exhibit *prudence*, which is the case for CRRA and most standard preferences),  $\Gamma$  is strictly increasing in consumption inequality. Since  $\Gamma$  is a linear combination of convex functions, it is itself convex.  $\Gamma$  attains its minimum at  $C = 0.5$ , when consumption inequality is minimal. Suppose now that this is the case at time  $t$ , i.e.  $C_t = 0.5$ . Then consumption inequality can only increase in the future, and  $\Gamma$  prices future payoffs highly compared to current ones in (13). Vice versa, a maximally unequal distribution of consumption today leads to low pricing of future payoffs. So when neglecting the constraints, we would expect asset prices to be *inversely*-U-shaped in  $A$ , which is the opposite of what occurs in equilibrium (we will later see that taking into account the constraints will overturn this result).

In order to quantify the pricing implications of  $\Gamma$  we now construct a hypothetical price  $Q_t \equiv \mathbb{E}_t \int_0^\infty e^{-\rho t} q \Gamma_s ds / \Gamma_t$ , and compare it with the equilibrium asset price  $P_t$ . Figure 2 shows  $Q$  (the thick lines) and  $P$  (the thin lines) with and without short-selling. When the wealth distribution is balanced,  $\Gamma_t$  prices the asset approximately correctly. This happens because the possibility of hitting the constraint is remote. Once we move closer to the constraint,  $\Gamma_t$  mis-prices the asset, and more so in the no-borrowing economy. Furthermore, we note that pricing future dividends by  $\Gamma_t$  alone can drive the asset price below the risk-neutral benchmark  $q/\rho$ , something that does not occur for the equilibrium pricing function.

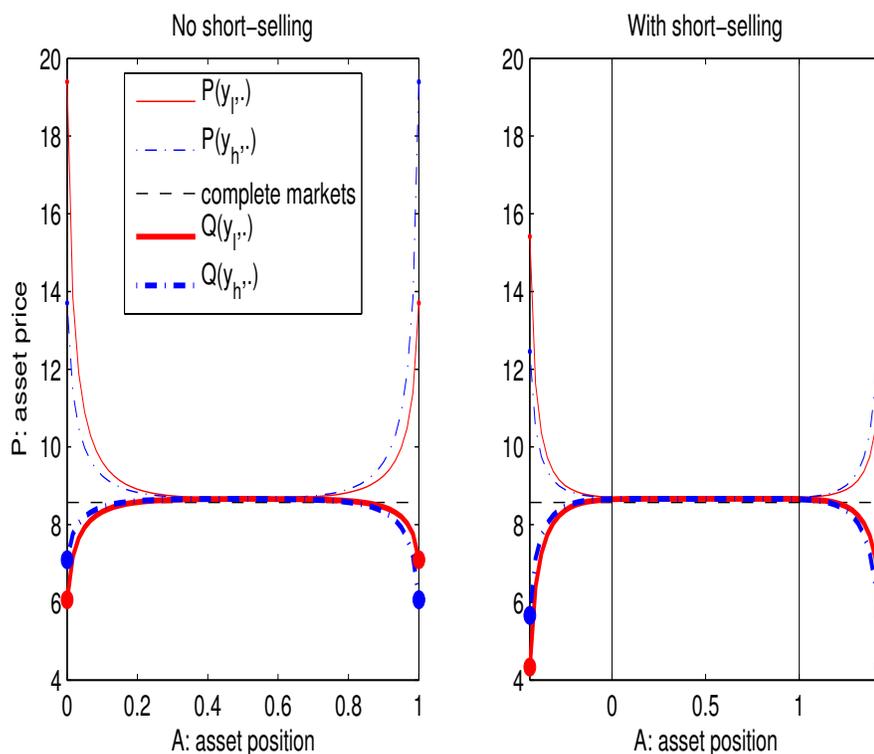


Figure 2: Disentangling the pricing kernel

Decomposition of the equilibrium asset price  $P$  (thin lines) into a hypothetical price  $Q_t \equiv \mathbb{E}_t \int_0^\infty e^{-\rho t} q \Gamma_s ds / \Gamma_t$  (thick lines), which only incorporates consumption inequality into the pricing kernel, and a remainder, which captures the importance of binding constraints. The short-selling limits are  $\bar{A} = 0$  (left panel) and  $\bar{A} = 0.44$  (right panel).

We now introduce an adjustment that tweaks the SDF at the constraints. Define

$$\Lambda_t \equiv \Gamma_t \xi^{i_t} \zeta^{j_t},$$

where the index  $i_t$  counts the number of times agent 1 has changed from being constrained to unconstrained until  $t$  (i.e. a reversal has occurred when  $A_t = -\bar{A}$  and  $y_t = y_l$  for  $\tilde{t} \in [0, t]$ ) and  $j_t$  counts the number of times that this has occurred to agent 2. The constants  $\xi$  and  $\zeta$  are defined as

$$\begin{aligned} \xi &\equiv \underbrace{\frac{u_c(1 - C(-\bar{A}, y_h))}{u_c(1 - C(-\bar{A}, y_l))}}_{>1} \underbrace{\frac{\frac{1}{2}u_c(C(-\bar{A}, y_l)) + \frac{1}{2}u_c(1 - C(-\bar{A}, y_l))}{\frac{1}{2}u_c(C(-\bar{A}, y_h)) + \frac{1}{2}u_c(1 - C(-\bar{A}, y_h))}}_{>1} > 1, \\ \zeta &\equiv \underbrace{\frac{u_c(C(1 + \bar{A}, y_l))}{u_c(C(1 + \bar{A}, y_h))}}_{>1} \underbrace{\frac{\frac{1}{2}u_c(C(1 + \bar{A}, y_h)) + \frac{1}{2}u_c(1 - C(1 + \bar{A}, y_h))}{\frac{1}{2}u_c(C(1 + \bar{A}, y_l)) + \frac{1}{2}u_c(1 - C(1 + \bar{A}, y_l))}}_{>1} > 1, \end{aligned}$$

and ensure that only the unconstrained agent prices the asset once the poor agent has dropped out of the market.  $\xi$  and  $\zeta$  are the product of two terms, both of which are strictly greater than unity. The first term encodes the unconstrained agent's marginal-utility ratio in case of a reversal; the second term is there to neutralize the pricing kernel  $\Gamma_t$ , which is only valid off the constraint.

We now state the recursive asset-pricing equation over a short  $\Delta t$  for the general case, i.e. for *any*  $A \in [-\bar{A}, 1 + \bar{A}]$ , i.e. including  $A \in \{-\bar{A}, 1 + \bar{A}\}$ :

$$P_t \Lambda_t = \mathbb{E}_t \int_t^{t+\Delta t} e^{-\rho(s-t)} \Lambda_t q ds + e^{-\rho \Delta t} \mathbb{E}_t [P_{t+\Delta t} \Lambda_{t+\Delta t}]. \quad (14)$$

When both agents are unconstrained, this equation may be obtained as Equation (13) by stochastically integrating over the average of the two Euler equations (8).<sup>15</sup> When a constraint binds, i.e. when  $A \in \{-\bar{A}, 1 + \bar{A}\}$  and the wealth-poor agent has low income, then Equation (14) also holds: as we take the limit  $\Delta t \rightarrow 0$  in (14), we obtain the unconstrained agent's Euler equation (9) or (10).

<sup>15</sup>Since the probability of  $i_t$  or  $j_t$  counting up becomes zero when  $\Delta t$  becomes small enough, this is valid for any interior  $A \in (-\bar{A}, 1 + \bar{A})$  for  $\Delta t$  small.

Integrating (14) stochastically over time and taking  $\Delta t \rightarrow \infty$  yields the asset-pricing equation

$$P_t \Lambda_t = \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} \Lambda_t q ds.$$

This equation says that if the constraint is currently binding or is likely to bind soon, then future dividends are valued highly, since  $i_t$  or  $j_t$  are expected to count up soon. This puts upward pressure on the price. For a fixed wealth level  $A$ , this effect is weaker when the economy is moving away from the constraint (i.e. when the asset-poor agent is income-rich), since then the probability of hitting a constraint is low. As we see in Figure 2, the upward price pressure from the constraint overrides the downward force exerted by  $\Gamma_t$  when wealth is highly concentrated. Hence, the terms  $\xi$  and  $\zeta$ , which capture binding constraints, are responsible for the U-shape of the pricing function  $P$  in equilibrium. Summarizing, asset prices are highest in times of high inequality because constraints are more likely to bind, not because consumption is unequal.

## 2.9 Infinite slope of $P(\cdot)$ and $C(\cdot)$ at binding constraint

A striking feature of equilibrium is that the pricing functions' slope becomes infinite when approaching the constraint in the case that the wealth-poor agent is also income-poor. Why is this? We will illustrate this for the case where no short-selling is allowed, but all arguments carry over to the general case as well.

By continuity of the consumption function  $C(\cdot, y_t)$  in  $A$ , the poor agent's consumption must converge to hand-to-mouth as he runs out of wealth, i.e.  $\lim_{A \rightarrow 0} C(A, y_t) = y_t$  (see Figure 3 for an illustration of the equilibrium variables over time as the constraint is hit). By the flow budget constraint (1) and the fact that the function  $P(\cdot)$  is bounded, this implies that he sells off the remaining assets ever slower, i.e.  $\dot{A} \rightarrow 0$  as  $A \rightarrow 0$  (see the two panels on the right). On the other hand, the buyers of the asset must be compensated for the risk of a price collapse in the case of an income reversal: there must be an appreciation of the asset in normal times. We see this in Equation (12), where  $\dot{P}/P$  converges to a positive constant as both  $C$  and  $P$  converge to constants as  $A \rightarrow 0$ . But since the economy drifts towards the constraint at ever lower speed, a meaningful appreciation of the asset is only possible

if the sensitivity of the asset price to the wealth distribution becomes infinite in the limit. Mathematically, since  $\dot{P} = P_A \dot{A}$  approaches a positive constant, it must be that  $P_A \rightarrow \infty$  since  $\dot{A} \rightarrow 0$ .<sup>16</sup>

By the same argument, we can establish that the slope of the consumption function  $C$  must approach  $-\infty$  at the constraint.<sup>17</sup>  $\dot{C} = C_A \dot{A}$  in Equation (11) approaches a negative constant – note that the numerator on the right-hand side is negative since  $C' > C$ . The fact that  $\dot{C} = C_A \dot{A}$  converges to a negative constant has two implications. First, the constraint must be reached in finite time, since  $C$  cannot become negative. Second, it implies  $C_A(A, y_l) \rightarrow \infty$  as  $A \rightarrow 0$ , since  $\dot{A} \rightarrow 0$ . Intuitively, if consumption is to decrease meaningfully over time it has to become infinitely sensitive to the wealth distribution as the economy’s state moves slower and slower.

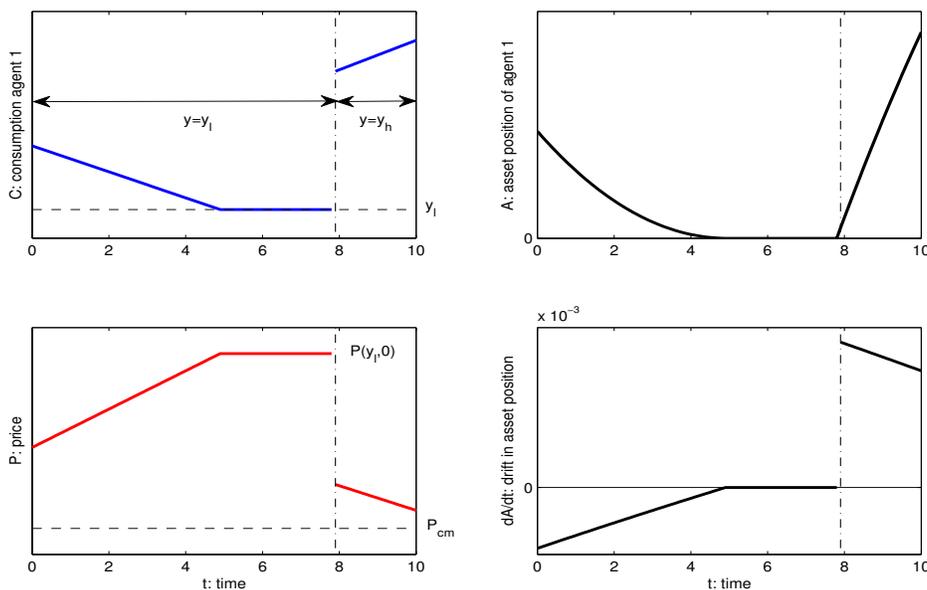


Figure 3: Hitting the constraint: a history

<sup>16</sup>It may not be immediately obvious that a well-behaved function exists that has the properties  $f(0) > 0$  and  $f'(0) = -\infty$ . A simple example for such a function is  $f(x) = C - \sqrt{x}$ .

<sup>17</sup>It may not be obvious in the lower panels in Figure 1 that the slope of  $C$  approaches infinity. However, this property becomes visible when zooming into the relevant region using a fine grid for  $A$ .

### 3 Bond economy

Is the relationship between inequality and asset prices that we found in the previous section peculiar to a Lucas-tree economy, or does it also hold for other types of assets? In the following two sections, we replace the Lucas tree first by a short-term bond and then by capital in a production economy. We find that asset returns in these economies behave very similar to the Lucas-tree economy.

#### 3.1 Solving the model

We first replace the tree with a short-term bond; otherwise all features of the environment are as in Section 2. Agents can buy bonds  $b_t \geq -\bar{B}$  at each  $t$ , where  $\bar{B} \geq 0$  is an exogenous borrowing limit. The bond costs  $1 - q_t \Delta t$  at  $t$  and yields one unit of consumption at  $t + \Delta t$ ;  $q_t$  is the bond yield. Agent 1's flow budget constraint is

$$c_t + \dot{b}_t = y_t + q_t b_t, \quad (15)$$

i.e. consumption and new bond purchases have to be financed by labor earnings and interest income. Labor income follows the same process as before, but we now normalize  $y_l + y_h = 1$  since there is no capital income in the economy. We assume utility to be of the CRRA form; this assumption is not essential for most of our results but simplifies the mathematics.

Using the same steps as in the Lucas-tree economy, the Euler equations for agents 1 and 2 are obtained as

$$\frac{\mathcal{A}u_c(C)}{u_c(C)} = \eta \left[ \frac{C'^{-\gamma}}{C^{-\gamma}} - 1 \right] - \alpha(C)\dot{C} = \rho - q, \quad (16)$$

$$\frac{\mathcal{A}u_c(1-C)}{u_c(1-C)} = \eta \left[ \frac{(1-C')^{-\gamma}}{(1-C)^{-\gamma}} - 1 \right] + \alpha(1-C)\dot{C} = \rho - q. \quad (17)$$

They say that marginal utility has to grow at the time discount rate minus the rate of return of the bond. The change in marginal utility again consists of two terms: the first represents the jump in consumption when a reversal occurs, and the second captures the effects consumption growth in normal times. At this point, we establish again the complete-markets benchmark, for which the same Euler equations have to hold. Since equilibrium consump-

tion is constant under complete markets, the left-hand sides of (16) and (17) are zero and we find the well-known benchmark  $q_{cm} = \rho$ .

In order to solve for the equilibrium in the incomplete-markets economy, we eliminate the bond yield as follows: multiply Equation (16) by  $C$  and Equation (17) by  $(1 - C)$  and subtract them from each other to obtain

$$\dot{C} = \frac{\eta}{\gamma} \left( \frac{1}{C} + \frac{1}{1-C} \right)^{-1} \left[ \frac{C'^{-\gamma}}{C^{-\gamma}} - \frac{(1-C')^{-\gamma}}{(1-C)^{-\gamma}} \right]. \quad (18)$$

This equation says that agent 1's consumption increases in normal times ( $\dot{C} > 0$ ) if and only if he faces downward consumption risk upon a reversal ( $C' < C$ ). This is the martingale property of consumption that we already obtained in the asset economy.

We now multiply Equation (16) by  $C$  and Equation (17) by  $(1 - C)$  and add them up to obtain an easy-to-interpret expression for the bond yield, which holds when neither of the agents is constrained:

$$q = \rho - \eta \left[ C \left( \frac{C'}{C} \right)^{-\gamma} + (1-C) \left( \frac{1-C'}{1-C} \right)^{-\gamma} - 1 \right] \quad \text{for all } \begin{cases} (y_l, B) : & B \in (-\bar{B}, \bar{B}], \\ (y_h, B) : & B \in [-\bar{B}, \bar{B}). \end{cases} \quad (19)$$

This equation shows that the bond return equals the rate of time preference minus a correction for consumption risk. This correction factor is a convex combination of both agents' consumption risk upon reversal, weighted by the agents' consumption levels and scaled by the reversal hazard  $\eta$ .

Consider now the situation where the first agent is constrained; again we guess this to be the case only if  $y = y_l$  at  $B = -\bar{B}$ . Agent 1 being constrained implies  $\dot{C} = 0$  and thus agent 2's Euler equation (17) implies that the bond yield in the constrained case is

$$q_{constr} = \rho - \eta \left[ \left( \frac{1-C}{1-C'} \right)^{\gamma} - 1 \right] \quad \text{for } (y, B) = (y_l, -\bar{B}, y_l). \quad (20)$$

A similar equation holds for the case when  $y = y_h$  at  $B = \bar{B}$ .

To solve for the equilibrium, we note that Equation (18) implies two ODEs for the func-

tions  $C(y_l, \cdot)$  and  $C(y_h, \cdot)$  with domain  $B \in [-\bar{B}, \bar{B}]$ . The boundary conditions come again from optimality at the constraint. Agent 1 being constrained implies the boundary condition  $C(y_l, -\bar{B}) = y_l - q_{constr}\bar{B}$ , where  $q_{constr}$  is given by (20). Agent 2 being constrained implies the second boundary condition,  $C(y_h, \bar{B}) = y_h + q_{constr}\bar{B}$ . We use a shooting algorithm: we guess  $C(y_h, -\bar{B})$ , then solve for consumption functions using the left boundary condition and the ODEs and finally check if right boundary condition is met. Having solved for  $C$ , we can then back out  $q$  from (19) and (20).<sup>18</sup>

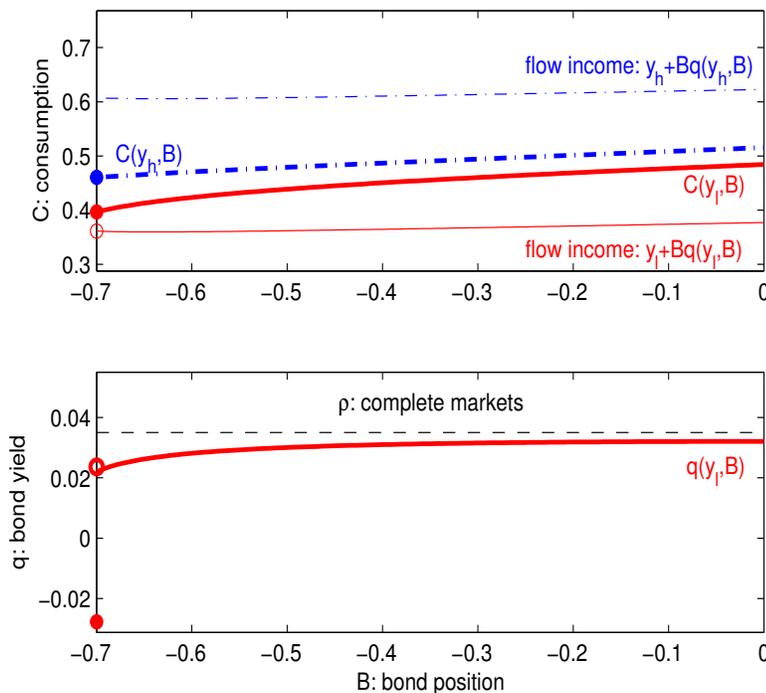


Figure 4: Bond economy

Parameters:  $\rho = 0.035$ ,  $\gamma = 2$ ,  $\eta = 0.25$ ,  $y_l = 0.377$ ,  $y_h = 0.623$  and  $\bar{B} = 0.7$ . The bottom panel only shows the bond yield for the low income realization,  $q(y_l, \cdot)$ . The pricing function for the high income realization,  $q(y_h, \cdot)$ , lies almost on top of  $q(y_l, \cdot)$  for all  $B \in (-\bar{B}, 0]$  but is continuous at  $B = -\bar{B}$ .

We parameterize the economy in the same way in which we parameterized the asset economy (see Section 2.6). Following Den Haan (2001), we let agents borrow up to slightly

<sup>18</sup>Since the economy is symmetric, we may also use the boundary condition  $C(y_l, 0) = 1 - C(y_h, 0)$  and stop solving the ODEs at  $B = 0$ , which is what we do in our algorithm.

more than once yearly labor earnings in the high-income state and set  $\bar{B} = 0.7$ . Figure 4 shows the equilibrium properties of the bond economy for the case that agent 1 is in debt (the region where agent 2 is in debt is entirely symmetric). The equilibrium consumption functions are as in the Lucas-tree economy: agents save in the high-income state and dissave in the low-income state; borrowing constraints are binding when wealth is exhausted and income is low; consumption functions are strictly increasing in income and wealth.

### 3.2 Properties of bond yields

Using Equations (19) and (20), one can easily show that bond yields have the following properties, which are also apparent in Figure 4:<sup>19</sup>

1.  $q \leq \rho = q_{cm}$ : bond yields never exceed the complete-markets benchmark.
2. The larger consumption risk  $|C - C'|$ , the lower the bond yield  $q$ .
3.  $q_{constr} < \lim_{B \rightarrow -\bar{B}} q(y_l, B)$ : bond yields drop discontinuously when one of the agents becomes constrained.

Similar to the asset economy, there are two forces that press bond yields below the complete-markets benchmark: consumption risk and binding constraints. On the interior of the state space, only consumption risk is at work. From Equation (19), we see that consumption risk lowers the bond yield since marginal utility is convex, a result which is reminiscent of Huggett (1993). Binding constraints lower the bond yield even further. Just

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<sup>19</sup>In order to prove Points 1 and 2 formally, we now show that  $q(C, \cdot)$  as defined by Equation (19) is a concave function in  $C'$  that attains its maximum at  $C' = C$  assuming the value  $q = \rho$ . Observe that when fixing  $C$  in Equation (19), the function  $h^{(C)}(C') \equiv q(C, C')$  is strictly concave and maximized at  $C' = C$  attaining the value  $q = \rho$ . This may be seen taking the derivatives

$$\frac{\partial h^{(C)}}{\partial C'} = \eta\gamma \left[ \left( \frac{C}{C'} \right)^{1+\gamma} - \left( \frac{1-C}{1-C'} \right)^{1+\gamma} \right],$$

$$\frac{\partial^2 h^{(C)}}{\partial C'^2} = -\eta\gamma(1+\gamma) \left[ \frac{C^{1+\gamma}}{C'^{2+\gamma}} + \frac{(1-C)^{1+\gamma}}{(1-C')^{2+\gamma}} \right].$$

Note that  $\partial^2 h^{(C)} / \partial C'^2 < 0$  for all  $(C, C') \in (0, 1)^2$  and that  $\partial h^{(C)} / \partial C' = 0$  for  $C' = C$ , which implies Points 1 and 2. Point 3 follows directly from comparing Equations (19) and (20): since the optimal consumption functions have to be continuous and the unconstrained agent faces downward consumption risk, it follows that  $q_{constr} < \lim_{B \rightarrow -\bar{B}} q(y_l, B)$ .

prior to group 1 becoming constrained the yield is determined by a weighted average of both groups' consumption risk, see Equation (19). Once group 1 becomes constrained, however, their supply of new bonds breaks away and only the unconstrained group prices the bond, see Equation (20). But the unconstrained group solely faces downside consumption risk and thus has a high demand for insurance, which makes the bond yield drop precipitously. The bond yield is thus discontinuous, both as a function of time (in the moment a constraint is hit) and as a function of the state,  $B$ .

To summarize, (i) yields are lowest when wealth and consumption inequality are maximal, and (ii) bond yields are most volatile – they actually display jumps – when an agent reaches or leaves the constraint. This is the same as for expected returns of the Lucas tree in the asset economy.

### 3.3 Discontinuity of $q$

The reader may wonder how the discontinuity of bond yields can be consistent with optimizing behavior in equilibrium, specifically with continuity of the consumption functions. To understand this, first note that  $C(y_l, \cdot)$  must be continuous at  $B = -\bar{B}$  by optimality of the consumption path. From agent 1's budget constraint (15), we see that a continuous consumption path over time requires

$$\underbrace{y_l - q_{constr}\bar{B}}_{=C(y_l, -\bar{B})} = \underbrace{y_l - q_{l,lim}\bar{B} - \dot{B}_{l,lim}}_{=\lim_{B \rightarrow -\bar{B}} C(y_l, B)}$$

where  $q_{l,lim}$  is the limit of  $q$  in (19) as  $B \rightarrow -\bar{B}$  for  $y = y_l$ , and  $\dot{B}_{l,lim}$  is the limit of  $\dot{B}$  as  $B \rightarrow -\bar{B}$  for  $y = y_l$ . Now, since  $q_{constr} < q_{l,lim}$ , it must be that  $\dot{B}_{l,lim} < 0$ , i.e. agent 1 obtains strictly positive revenues from selling new bonds in the moment before hitting the constraint.<sup>20</sup> The economics are the following: agent 1's interest payments drop in the moment that he becomes constrained because the interest rate tanks. This fall in interest

<sup>20</sup>Note that here, we allow for a slightly extended notion of differential equations, in the sense that we allow  $\dot{B}$  to be a discontinuous function of the state (and thus of time) at  $B = -\bar{B}$ . This is not problematic in the sense that we can solve for the trajectory of  $B$  inside the interval  $(-\bar{B}, \bar{B})$  as an ODE, and once the trajectory hits the point  $-\bar{B}$  the economy just stays put at this point until a reversal occurs. It is interesting to observe that we have  $\dot{B}_{l,lim} < 0$  in the bond economy, while in the asset economy asset sales tend to zero at the constraint, i.e.  $\lim_{A \rightarrow -\bar{A}} \dot{A} = 0$ .

payments is exactly offset by agent 1 not issuing new bonds any more, exactly leveling out his consumption path.

For the wealthy agent, the picture is as follows. Before the constraint is reached, relatively high interest rates make it optimal to save and buy strictly positive quantities of new bonds. Once the constraint is reached, however, equilibrium in the bond market requires that agent 2 does not further increase her bond position. This is ensured by the interest rate dropping just enough to make maintaining a constant bond position optimal for agent 2. The wealthy agent's consumption path is continuous since the discontinuous decrease in interest revenue at  $B = -\bar{B}$  is exactly offset by expenditures for new bonds dropping down to zero upon reaching the boundary  $B = -\bar{B}$ .

In our calibration, the bond yield even drops to a negative value in this situation. Since there is no storage in our economy, this is not at odds with optimal behavior of agents. Also in reality, negative real interest rates do actually occur in situations with close-to-zero nominal interest rates and positive inflation.

## 4 Capital economy

In the asset and bond economies, the supply of the security is fixed so the security's price has to adjust in order to clear markets. We will now see that asset returns display very similar properties in a production economy where the price of the capital is fixed and all adjustments occur through quantities.

### 4.1 Setting and solution

Consider a standard neo-classical production economy. The sole consumption good is produced using a Cobb-Douglas production function:  $Y = K_{agg}^\alpha N^{1-\alpha}$ , where  $K_{agg}$  is the aggregate capital stock, aggregate labor  $N$  is normalized to 1, and where  $\alpha \in (0, 1)$ . Capital is the only asset, and its accumulation is standard. We impose a non-negativity constraint on agents' capital holdings  $k$ . Agent 1's labor endowment  $z \in \{z_l, z_h\}$ ,  $z_l \leq z_h$ , follows a two-state Poisson process with switching hazard  $\eta$ . It is again perfectly negatively correlated with agent 2's endowment process, which is given by  $1 - z$ . In contrast to before,

wages are now endogenous and depend on the capital-labor ratio. The equilibrium interest rate equals the marginal product of capital minus the depreciation rate  $\delta$ , and is thus decreasing in the aggregate capital stock.

Since the analysis for the capital economy differs somewhat from the asset and bond economies, we now describe how we solve for equilibrium. The aggregate state of the economy is given by  $(z, K, \tilde{K})$ , where  $K$  and  $\tilde{K}$  are the amounts of capital owned by group 1 and group 2, respectively. Again, we have to allow an individual agent to deviate from the group position when stating his problem. We denote the capital positions of an individual from group 1 and group 2 by  $k$  and  $\tilde{k}$ , respectively. The agents take as given the perceived laws of motion  $\dot{K}$  and  $\dot{\tilde{K}}$ . Agent 1's individual state is  $(z, K, \tilde{K}; k)$ , and his HJB is

$$\begin{aligned} \rho V &= \max_c \{u(c) + \dot{k}V_k\} + \dot{K}V_K + \dot{\tilde{K}}V_{\tilde{K}} + \eta V_z \\ \text{s.t.} \quad \dot{k} &= [r(K + \tilde{K}) - \delta]k + zw(K + \tilde{K}) - c. \end{aligned}$$

where the constraint is agent 1's flow budget constraint. We use again the "discrete derivative"  $V_z \equiv V(z', \cdot) - V(z, \cdot)$ , and we make it explicit that the rental rate of capital  $r$  and the wage rate  $w$  depend on the aggregate capital stock  $K_{agg} = K + \tilde{K}$ . In equilibrium, each factor is paid its marginal product, thus  $r = \alpha K_{agg}^{\alpha-1}$  and  $w = (1 - \alpha)K_{agg}^\alpha$ . The FOC for consumption is  $u_c(c^*) = V_k$ .

When agent 1 is unconstrained, his consumption process  $c$  must fulfill an Euler equation, which is obtained by differentiating the HJB in  $k$ :

$$\rho V_k = u_c(c^*)c_k^* + \frac{d\dot{k}}{dk}V_k + \dot{k}V_{kk} + \dot{K}V_{kK} + \dot{\tilde{K}}V_{k\tilde{K}} + \eta V_{kz}.$$

At this step we have used  $d\dot{K}/dk = d\dot{\tilde{K}}/dk = 0$ , which holds since the agent is atomistic and cannot affect aggregate conditions. Now, we use  $d\dot{k}/dk = r - \delta - c_k^*$  and the FOC  $u_c(c^*) = V_k$  to find

$$(\rho + \delta - r)V_k = \underbrace{\eta V_{kz} + \dot{K}V_{kK} + \dot{\tilde{K}}V_{k\tilde{K}} + \dot{k}V_{kk}}_{=AV_k}.$$

We recognize the infinitesimal generator applied to  $V_k$  on the right-hand side. In order to obtain the standard Euler equation, we use again the FOC and impose that agent 1 does not deviate from the aggregate consumption behavior of his group in equilibrium, i.e. we set  $c = C$ :

$$\frac{\mathcal{A}u_c(C)}{u_c(C)} = \underbrace{\eta \left[ \frac{u_c(C')}{u_c(C)} - 1 \right]}_{\text{jump in } z} + \underbrace{\frac{u_{cc}(C)}{u_c(C)} \dot{C}}_{\% \text{ change in } u_c} = \rho + \delta - r, \quad (21)$$

where we denote by  $\dot{C} = C_K \dot{K} + C_{\tilde{K}} \dot{\tilde{K}}$  the infinitesimal change in consumption in normal times and where  $r$  is a function of the aggregate capital stock  $K + \tilde{K}$ . This Euler equation is entirely standard and says that marginal utility should grow at the expected rate  $\rho$  minus the interest rate  $r - \delta$ . The change in marginal utility can be decomposed into consumption risk upon an income reversal and consumption growth in normal times.

At this point, we again take a detour and analyze the complete-markets benchmark. We restrict ourselves to the steady state for aggregate capital here. Since complete markets provide perfect consumption insurance, in steady state the left-hand side of the Euler equation (21) is zero and we obtain

$$r_{ss} = \rho + \delta, \quad K_{ss} = \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}.$$

In order to solve for equilibrium in the incomplete-markets economy, we follow the same procedure as for the asset economy and backward-iterate on policy functions in a finite-horizon setting until convergence. From the Euler equation (21), we back out the time change in consumption in normal times as follows:

$$\dot{C} = \frac{r - \delta - \rho}{\alpha(C)} + \frac{\eta}{\alpha(C)} \left[ \frac{u_c(C')}{u_c(C)} - 1 \right], \quad (22)$$

where  $r = \alpha(K + \tilde{K})^{\alpha-1}$  is a function of the aggregate capital stock. In the time-dependent case an additional time derivative shows up, i.e. we have  $\dot{C} = C_t + C_K \dot{K} + C_{\tilde{K}} \dot{\tilde{K}}$  on the left-hand side of (22). As before, we denote by  $\alpha(C) = -u_{cc}(C)/u_c(C)$  the coefficient of absolute risk aversion.

It is worthwhile to pause here for a moment in order to understand the saver's incentives.

The first term on the right-hand side of (22) says that if the interest rate  $(r - \delta)$  is high, then consumption grows rapidly – a high marginal product of capital provides incentives to save. As agents accumulate more capital, the interest rate is driven down which discourages savings, just as in a representative-agent economy. The discount rate  $\rho$  has the opposite effect.

The second term on the right-hand side of (22) says that consumption growth is high in normal times if there is downside consumption risk upon an income reversal. This term encodes the precautionary motive: downward consumption risk induces higher savings. Another reading of this term is again that the optimal consumption process has a (modified) martingale property. As in the other economies, the Euler equation and thus (22) may hold with inequality when an agent has zero capital and is constrained. Again, we find that this only occurs when an agent has the low income realization.

Similar to the asset economy, we solve for equilibrium using standard PDE techniques: we backward-iterate on the consumption rules for groups 1 and 2 until convergence, backing out  $C_t$  and  $\tilde{C}_t$  from Equation (22) for each aggregate state  $(z, K, \tilde{K}, t)$ .

## 4.2 Equilibrium analysis: the ergodic set

Again, we follow the parameterization from the asset economy (see Section 2.6) as closely as possible and make standard assumptions on the parameters specific to this economy, see the note below Figure 5 for details.

In order to capture the wealth distribution in a single variable as in the two economies before, we map the state variables  $(K, \tilde{K})$  to

$$P = \frac{K}{K + \tilde{K}}, \quad K_{agg} = K + \tilde{K}.$$

Thus  $P \in [0, 1]$  is the fraction of total wealth owned by group 1 and is akin to the variables  $A$  and  $B$  in the other economies. This new pair of state variables is used in the phase diagram in Figure 5, which fully characterizes the capital economy's dynamics.

The arrows portray the joint evolution of the wealth distribution and the aggregate capital stock: the red solid arrows are relevant when group 1 has low productivity, and the blue dashed arrows describe the dynamics when group 1 has high productivity. As a reference

point, the steady-state capital stock of the corresponding representative-agent economy is given by the dashed horizontal line. The ergodic set – the set of states the economy stays in in the long run – is the bra-shaped shaded area.

We now show how to find the ergodic set, and explain the equilibrium’s properties along the way. First, note that there are exactly two points – the circles in the top left and right of the graph – in which the economy is at a temporary steady state: it does not leave this state unless an income reversal occurs. If agent 1 stays with the low income for a long time, the economy will converge to the red circle from any initial condition, as the red solid arrows show. When agent 1 receives the high income for a long time, however, then the blue dashed arrows take the economy to the blue circle. Thus, these two points must belong to the ergodic set.

Consider now the situation at the red circle. Group 1 has low productivity and zero wealth, whereas group 2 is productive and owns the entire capital stock. The aggregate capital stock reaches its maximal level within the ergodic set at this point; the interest rate is driven down to the point where it is not attractive for agent 2 to accumulate further precautionary savings any more.

Once an income reversal occurs at the red circle, the dynamics are given by the blue dashed arrows, and the economy follows the upper envelope of the ergodic set south-east. Group 1 is now accumulating capital, but group 2 is driving down its buffer stock of savings at a faster pace, thus diminishing aggregate capital and boosting the interest rate. Indeed, group 2 dissaves much more than group 1 saves at this point; capital decumulation is maximal within the ergodic set, which manifests itself in the downward component of the blue dashed phase vector being maximal here. This also means that the interest rate changes fastest in this situation, which is reminiscent of asset returns being most volatile in situations of extreme inequality in the asset and bond economies. Unlike in the other economies, however, returns decrease in a *continuous* fashion in the capital economy.<sup>21</sup>

Following down the upper envelope of the ergodic set from the red circle, we see that if another reversal occurred on this downward path, the red solid arrows would take the economy south-west towards the interior of the ergodic set. When following the upper envelope of the ergodic set farther, however, we will eventually reach the V-shaped kink in the

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<sup>21</sup>The interest rate is a continuous function of aggregate capital, which in turn is a continuous function of time even when a reversal occurs.

ergodic set's boundary at  $P = 0.5$  and  $K \simeq 3.4$ . By symmetry, the two upper envelopes of the ergodic set must meet here. At this point, the wealth distribution is perfectly balanced and both agents have reasonable buffer stocks of savings. The desire to save is comparatively low, and for either income realization the aggregate capital stock diminishes, as the downward-pointing pair of phase vectors indicates. Interest rates are thus low and decreasing, and their time variation is low (as can be gleaned from the downward component of phase vectors being small).

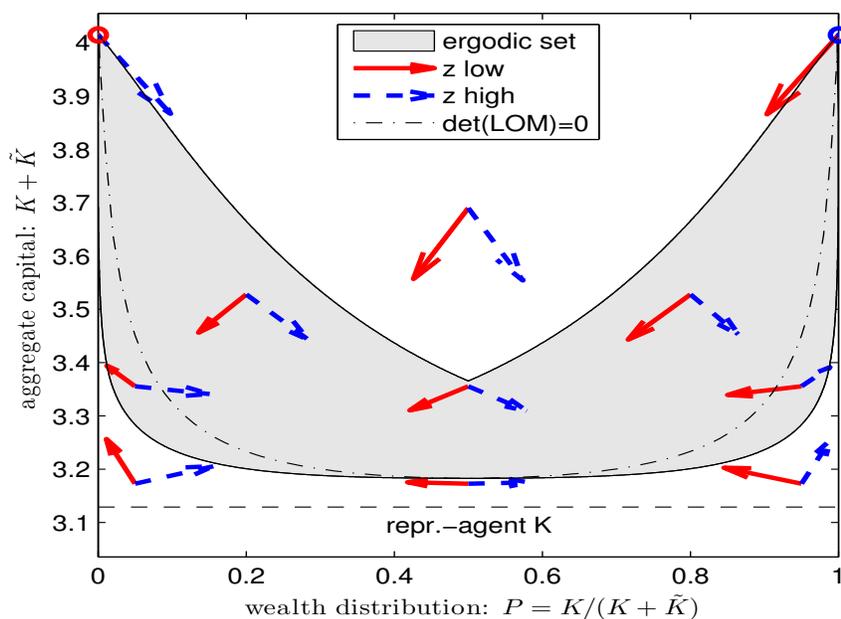


Figure 5: Phase diagram for the capital economy

Parameters: CRRA utility with  $\gamma = 2$  and  $\rho = 0.035$ ; production function  $Y = K^\alpha L^{1-\alpha}$  with  $\alpha = 0.3$  and depreciation  $\delta = 0.1$ ; labor-productivity process  $z_l = 0.377$ ,  $z_h = 0.623$  with switching hazard  $\eta = 0.25$ . The dash-dotted line marks points where the determinant of the matrix of phase vectors is zero, i.e. where they stand to each other in an angle of 180 or 0 degrees.

If there is a string of income reversals in quick succession parting from the V-shaped kink, then the economy see-saws down like a falling leaf to the bottom of the ergodic set, where both phase arrows become horizontal. We have now reached what will turn out to be the lower boundary of the ergodic set at  $P = 0.5$  and  $K \simeq 3.2$ . After this history,

which is one of maximal income equality, the minimal aggregate capital stock and thus the maximal interest rate within the ergodic set are realized. The capital stock stops short of the representative-agent benchmark, though, due to the presence of the precautionary motive. Just as in the asset and bond economies, the variability of interest rates over time is minimal (indeed zero) in this situation of maximal equality.<sup>22</sup>

From this bottom point, following the red solid arrows we can now trace the ergodic set's lower envelope towards the left by considering a long spell of low productivity for group 1. Along this trajectory, agent 1 is consuming up his buffer stock of capital, his consumption falling as he approaches the constraint. Agent 2, however, is accumulating ever larger wealth. Since agent 1 is decumulating capital ever slower and agent 2 is increasing savings as he gets richer, aggregate capital increases, and interest rates fall, as we approach  $P = 0$ . Again, if an income reversal occurred while tracing out this lower envelope, the blue dashed arrows would take us towards the interior of the ergodic set. Eventually, and at a decreasing speed, the economy reaches the starting point of our voyage, the red circle. At this point, group 2 stops accumulating capital since the interest rate has decreased sufficiently to make additional savings unattractive.

In the upper-left corner of the ergodic set, we note that an income reversal leads to a sharp turnaround in the trend of capital accumulation, and thus in the trend of interest rates: the red solid arrows point upward, whereas the blue dashed arrows point sharply downward. Thus the time-series path of the interest rates is more erratic and harder to predict in times of extreme wealth inequality. This is in line with returns in the asset and bond economies being most volatile close to the constraints.

Since the right-hand side of the graph is symmetric to the left-hand side, this concludes the construction of the ergodic set: the set cannot be left, any point in its interior can be reached by concatenating suitable trajectories departing from its boundaries, and any trajectory starting outside the set must enter the set eventually, as inspection of the phase diagram confirms.

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<sup>22</sup>At this lowest point ( $P = 0.5$  and  $K \simeq 3.2$ ), the ergodic set's boundary is touched by the dash-dotted line. This line marks the locus where the determinant of the matrix of phase vectors is zero, i.e. where they stand to each other in an angle of 180 (or 0) degrees. This line demarcates the upper area of the state space, where the quiver of phase arrows is pointing downward, from the lower area, where the quiver is pointing upward. Evaluating this determinant on the boundaries of the ergodic set formally confirms that we have indeed found the ergodic by our construction.

### 4.3 Summary of results

To sum up, our capital economy is an example of an economy in which output fluctuations are caused solely by re-distributional shocks.<sup>23</sup> Since productivity shocks are correlated across individuals, they do not wash out at the aggregate level as idiosyncratic shocks do in Aiyagari's (1994) economy. This causes the aggregate demand for precautionary savings to vary. Wealth-rich agents have a larger influence on aggregate capital than the wealth-poor do, thus the concentration of wealth matters for aggregate savings.

Second, we note that these fluctuations in the capital stock constitute an inefficiency. There is over-accumulation of capital in times of high wealth inequality because rich agents cannot write insurance contracts against adverse income shocks with poor agents; instead, the rich accumulate more capital in order to self-insure. This over-accumulation of capital in times of high wealth inequality comes on top of the fact that the capital stock is in general too high in incomplete-markets economies, which is well-known from Aiyagari's (1994)'s economy with idiosyncratic shocks.

As for the return to capital, it reaches its lowest level and maximal (time) variability when wealth is concentrated in the hands of one group. This is in line with the predictions of the asset and the bond economy.

We conclude that the *qualitative* properties of asset returns in our class of incomplete-markets economies do not depend on the nature of the security that is traded. In the following section we will have a brief look at the *quantitative* predictions that our three theories have for asset returns.

## 5 Gauging the quantitative implications

Is the interplay between inequality and asset prices that our three model economies describe of any quantitative relevance? This section makes a first, rough attempt to gauge the magnitude of swings in asset returns that our theories predict, and finds that they could be large enough to warrant further research.

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<sup>23</sup>If markets were complete, the capital stock and thus output would not fluctuate once the steady state is reached, as is well-known.

Economy	(expected) return	asset price	capital stock
Asset	[-5.80, 3.46]	[8.68, 19.40]	–
Bond	[-2.77, 3.21]	–	–
Capital	[1.34, 3.34]	–	[3.18, 4.01]
Complete markets	3.50	8.57	3.13

Table 1: Range of asset returns in % and other variables in numerical example

Parameters (all economies):  $\rho = 0.035$ ,  $u(c) = c^{1-\gamma}/(1-\gamma)$  with  $\gamma = 2$ ,  $\eta = 0.25$ ,  $y_h/y_l = z_h/z_l = 1.2456/0.7544$ . Asset and capital economy:  $q = \alpha = 0.3$ . Bond economy:  $\bar{B} = 0.7$ . Capital economy:  $\delta = 0.1$ , no borrowing. We calculate the *expected* return in the asset economy since the asset is not safe. Complete-markets benchmark: the expected return of all assets is  $\rho = 3.5\%$  in all economies; the numbers in Columns 3 and 4 refer to the complete-markets version of the asset ( $P_{cm}$ ) and capital economy ( $K_{ss}$ ), respectively.

For all three economies, we stick to the same parameters as before; we rule out short-selling in the asset economy and set  $\bar{A} = 0$ . Table 1 shows the range that several variables of interest cover in equilibrium and compares them to the complete-markets benchmark. We see that the asset economy displays the widest range of asset returns, followed by the bond economy. The capital economy displays somewhat lower variations in returns, but the variability of the aggregate capital stock is substantial, amounting to around 20% of its steady-state level. The asset economy generates most variation in returns, and also large swings in the level of asset prices. We conclude that the range of time variation in economic outcomes is of a quantitatively significant magnitude in all three economies.

## 6 Conclusion

We have studied a class of dynamic incomplete-markets economies in which shocks are not idiosyncratic but specific to entire groups of agents. Examples for such shocks are changes to the college premium, redistributive tax reform and shocks to industries, occupations, regions and entire countries. In these settings, the wealth distribution becomes a state variable and helps to predict asset prices. We find that a high concentration of wealth goes along

with high asset prices, low expected returns and high volatility of prices and returns. A first quantitative pass suggests that these effects can be substantial, which warrants further study of this channel.

Since empirical work finds most labor-income risk to be idiosyncratic, more realistic extensions of our model would require combining group-specific shocks with idiosyncratic ones. This is a monumental task, however. The wealth distribution – an infinite-dimensional object – becomes a state variable in such an environment, and our analysis suggests that one should not neglect this distribution’s predictive power for prices when there are group-specific shocks. On a positive note, our analysis may be able to guide the selection of moments in dimension-reduction techniques à la Krusell-Smith, which aim to distill the most important properties of the wealth distribution into a small set of moments. Our results suggest that it is important to keep track of the number of agents in a group that are at, or close to, a constraint: a large number of agents being concentrated in such regions should herald swings in asset demand that predictively change asset returns.

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